PHYS 271 Formula Sheet

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Constants

$h = 6.6261 \times 10^{-34}\,\text{J s} = 4.135 \times 10^{-15}\,\text{eV s}$

$\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34}\,\text{J s} = 6.6582 \times 10^{-16}\,\text{eV s}$

$c = 3 \times 10^8\,\text{m/s}$ $\quad \epsilon_0 = 8.854 \times 10^{-12}\,\text{F/m}$ $\quad \mu_0 = 4\pi 10^{-7}\,\text{Wb/A m}$

$1\,\text{J} = \frac{1}{e}\,\text{eV}$ $\quad 1\,\text{J} = 2.39 \times 10^{-4}\,\text{kcal}$ $\quad e = 1.6022 \times 10^{-19}\,\text{C}$

$k = 1.38 \times 10^{-23}\,\text{J/K}$ $\quad R_H = 1.097 \times 10^7\,\text{m}^{-1}$ $\quad N_A = 6.022 \times 10^{23}$

$u = 1.661 \times 10^{-27}\,\text{kg} = 931.494\,\text{MeV/c}^2$

$m_e = 9.109 \times 10^{-31}\,\text{kg} = 0.511\,\text{MeV/c}^2$

$m_p = 1.673 \times 10^{-27}\,\text{kg} = 938.27\,\text{MeV/c}^2$

$m_n = 1.675 \times 10^{-27}\,\text{kg} = 939.57\,\text{MeV/c}^2$

$\mu_B = 9.274 \times 10^{-24}\,\text{J/T} = 5.788 \times 10^{-5}\,\text{eV/T}$

Special Relativity

Galilean transforms: $t' = t, \quad x' = x + vt, \quad y' = y, \quad z' = z$

Inertial frame: A reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.

Postulates:

1. The laws of physics are the same in all inertial frames of reference.
2. Speed of light same in all inertial frames.

Relativity of simultaneity: Two spatially-separated events simultaneous in one reference frame are not simultaneous in any other inertial frame moving relative to the first.

If events occur at same spatial location, simultaneous in all frames.

Time dilation (proper time): $\Delta\tau = \frac{\Delta t}{\gamma}$ $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}},\ \beta = \frac{v}{c}$

- Stationary frame experiences longer time relative to moving frame. $\Delta\tau$ is the shortest possible interval.
- $\Delta\tau$ is time interval btw two events is time interval measured by an observer for whom both events occur at the same location.

Length contraction (proper length): $L_0 = \gamma L$

- L_0 is length of obj measured by an observer at rest relative to it.
- L_0 is longest possible length.

Tip: $\Delta\tau$ and L_0 are measurements of S' by S

Approximation: If $\beta \ll 1,\ \gamma \approx 1 + \beta^2/2$ $\quad 1/\gamma \approx 1 - \beta^2/2$

Radioactive decay: $N(t) = N_0 e^{-t/\tau}$, $\quad$ account for time dilation

Lorentz transform

$$S \mapsto S': \quad ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z$$

$$S' \mapsto S: \quad ct = \gamma(ct' + \beta x'), \quad x = \gamma(x' + \beta ct'), \quad y = y', \quad z = z'$$

4-vectors: $\underset{\sim}{x} = [ct, x, y, z]^T = [ct, \mathbf{r}]^T$

$$\underset{\sim}{x}' = \Lambda \underset{\sim}{x}, \quad \Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

CAUTION: Account for the c in ct!

Transformations in Minkowski space

$x' = x\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ $\quad ct' = ct\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ $\quad \tan\alpha = \frac{v}{c}$

Intervals: $\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ $\quad$ (Lorentz invariant)

- Timelike: $(c\Delta t)^2 > \Delta x^2 + \Delta y^2 + \Delta z^2$ $\quad \Delta s^2 > 0$
- Lightlike: $(c\Delta t)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ $\quad \Delta s^2 = 0$
- Spacelike: $(c\Delta t)^2 < \Delta x^2 + \Delta y^2 + \Delta z^2$ $\quad \Delta s^2 < 0$

Causality: If events A and B are linked, then they have to be timelike or lightlike separated, and all observers agree on temporal order.

- If events timelike, their order is fixed (inside the light cone).
- If events lightlike, event A reaches B at exact instance B occurs. All observers agree that A occurs before B.
- If events spacelike, no causality. Not all observers agree on order (outside of the light cone).

Relativistic Doppler Effect

$f_{\text{obs}} = f_{\text{src}} \frac{\sqrt{1-\beta^2}}{1 + \beta\cos\theta}$, $\quad$ light moving at θ wrt observer

Directly away: $f_{\text{obs}} = f_{\text{src}}\sqrt{\frac{1-\beta}{1+\beta}}$

Directly towards: $f_{\text{obs}} = f_{\text{src}}\sqrt{\frac{1+\beta}{1-\beta}}$

Transverse: $f_{\text{obs}} = f_{\text{src}}\sqrt{1-\beta^2}$

Moving Charges

$\lambda_p - \lambda_n = \lambda\beta^2\gamma$ $\quad \lambda_p$ is +ve q density, λ_n is -ve q density

Assuming $\gamma \approx 1$: $\mathbf{F} = \frac{qv^2\lambda}{2\pi r}\mu_0\,\hat{k}$ $\quad \lambda_n' = \lambda/\gamma, \quad \lambda_p' = \lambda\gamma$

Highly relativistic case: $\mathbf{F}' = \gamma\mathbf{F}$ $\quad \mathbf{F}'_\perp = \gamma\mathbf{F}_\perp$ $\quad \mathbf{F}'_\parallel = \mathbf{F}_\parallel$

Relativistic Mechanics

$u_x = \frac{u_x' + v}{1 + vu_x'/c^2}$ $\quad u_y = \frac{u_y'}{\gamma(1 + vu_x'/c^2)}$ $\quad u_z = \frac{u_z'}{\gamma(1 + vu_x'/c^2)}$

$u_x' = \frac{u_x - v}{1 - vu_x/c^2}$ $\quad u_y' = \frac{u_y}{\gamma(1 - vu_x/c^2)}$ $\quad u_z' = \frac{u_z}{\gamma(1 - vu_x/c^2)}$

- u is velocity of event relative to S.
- v is velocity of S' relative to S.
- u' is velocity of event relative to S'.
- γ is wrt to v.

4-velocity: $\underset{\sim}{v} = [\gamma c, \gamma\dot{x}, \gamma\dot{y}, \gamma\dot{z}]^T = [\gamma c, \gamma\mathbf{u}]^T$ $\quad \underset{\sim}{u}' = \Lambda\underset{\sim}{u}$

4-momentum: $\underset{\sim}{p} = \gamma m\mathbf{u}$ $\quad$ Forces: $\mathbf{F} = \gamma m\frac{d\mathbf{u}}{dt}$

$E = \gamma mc^2 = E_k + E_0$ $\quad E_k = (\gamma - 1)mc^2$ $\quad E_0 = mc^2$

$\underset{\sim}{p} = \left[\frac{E}{c}, p_x, p_y, p_z\right]^T = \left[\frac{E}{c}, \mathbf{p}\right]^T$ $\quad E^2 = m^2c^4 + p^2c^2$

Invariant mass (Lorentz invariant): $\underset{\sim}{p} \cdot \underset{\sim}{p} = \frac{E^2}{c^2} - p^2 = m^2c^2$

For $p \gg m,\ E \approx pc$ Classical physics: $E = \frac{p^2}{2m}$

For $\mathbf{u} \perp \mathbf{B},\ quB = \gamma m \frac{u^2}{R} \implies \gamma = \frac{qBR}{mu}$ $p = qBR = \gamma mu$

Approach: Set up E-cons and p-cons → sub into E-p relationship.

General Relativity

Equivalence principle: A homogeneous gravitational field is completely equivalent to a uniformly accelerated reference frame.

Gravitational lensing: $\alpha = \frac{4GM}{c^2 R}$

Einstein's field eqs: $G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$

Charge Quantization

J.J. Thompson: discovered electrons ("cathode rays").

$$Q = Ne \implies W = N\left(\frac{1}{2}mu^2\right) = \frac{Q}{e}\left(\frac{1}{2}mu^2\right)$$

$$mu = QBR \implies \frac{e}{m} = \frac{2W}{QB^2R^2}$$

Millikan: oil drop expt to det e.

Stoke's law: $v_T = \frac{2gr^2}{9\eta}(\rho_{\text{particle}} - \rho_{\text{medium}})$

$$qE = mg \implies q = \frac{mg}{E} = \frac{4\pi r^3 \rho_{\text{oil}} g}{3E}$$

Energy Quantization

Wien's displacement law: $\lambda_{\max} T = 2.898 \times 10^{-3}\,\text{m K}$

Radiation power: $P(T) = \sigma A T^4$

Radiation intensity: $R(T) = \sigma T^4$

Stefan's constant: $\sigma = 5.6703 \times 10^{-8}\,\text{W}/(\text{m}^2\,\text{K}^4)$

$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$, cubic cavity of length L, $E = 0$ at walls:

Modes: $N = \frac{8\pi L^3}{3\lambda^3}$

Modes/λ: $\left|\frac{dN}{d\lambda}\right| = \frac{8\pi L^3}{\lambda^4} = \frac{8\pi V^3}{\lambda^4}$

Modes/$(\lambda \cdot V)$: $\rho_n(\lambda) = \frac{8\pi}{\lambda^4}$

Classical Blackbody Formula via Equipartition Thm

E density/λ: $u(\lambda) = \frac{8\pi}{\lambda^4}\langle E\rangle = \frac{8\pi kT}{\lambda^4}$ $k = 1.38 \times 10^{-23}\,\text{J/K}$

Rayleigh-Jeans eq: $R(\lambda) = \frac{2\pi kTc}{\lambda^4}$

UV catastrophe: $\rho_E(\lambda) = \int_0^\infty u(\lambda)\, d\lambda \to \infty$ (infinite E density)

Exptl obs: $\lambda \to 0 \implies u(\lambda) \to 0$ (Planck solves this issue)

Planck's Blackbody Formula

Energy: $\langle E\rangle = \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$

E density/λ: $u(\lambda) = \frac{8\pi}{\lambda^4}\langle E\rangle = \frac{8\pi}{\lambda^4}\frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$ $c = h\nu$

E density/ν: $u(\nu) = \frac{dE}{d\nu} = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$

Radiant intensity: $R(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\left(\frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}\right)$

λ small $\implies R(\lambda) = \frac{2\pi kTc}{\lambda^4}$ λ large $\implies R(\lambda) \to 0$

$$R \propto \frac{1}{\lambda^5\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]}$$

Photoelectric Effect

$E_k = eV \implies$ electron energy depends on ν of light, NOT intensity

$E_{k,\max} = h\nu - \phi$ $\phi = hf_c$ (set $E_k = 0$)

X-rays: $\lambda_{\min} = \frac{1.2407 \times 10^{-6}}{V}$

Compton Effect

$\underset{\sim}{p}_{\gamma,1} = \left[\frac{h}{\lambda_1}, \frac{h}{\lambda_1}, 0, 0\right]$ $\underset{\sim}{p}_{\gamma,2} = \left[\frac{h}{\lambda_2}, \frac{h}{\lambda_2}\cos\theta, \frac{h}{\lambda_2}\sin\theta, 0\right]$

$\underset{\sim}{p}_{e,1} = [m_e c, 0, 0, 0]$ $\underset{\sim}{p}_{e,2} = \left[\frac{E}{c}, p_e\cos\phi, -p_e\sin\phi, 0\right]$

$\lambda_2 - \lambda_1 = \underbrace{\frac{h}{m_e c}}_{\lambda_c}(1 - \cos\theta)$ $p_\gamma = \frac{E_\gamma}{c}$ $E_\gamma = h\nu = \frac{hc}{\lambda}$

Compton shift: $\Delta\lambda = \lambda_c(1 - \cos\theta)$

Rutherford Scattering

Impact parameter: $b = \frac{Qq_\alpha}{4\pi\epsilon_0 m_\alpha v^2}\cot\frac{\theta}{2} = \frac{Qq_\alpha}{4\pi\epsilon_0(2E_k)}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$

Area: $\sigma = \pi b^2 = \pi\left(\frac{Qq_\alpha}{8\pi\epsilon_0 E_k}\right)^2\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$

Scattering fraction: $f = \sigma nL$ nL : atoms per unit area

Atoms/unit V: $n = \frac{1000\rho N_A}{M}$ ρ (kg/m^3), M (u)

A causing α to scatter at angle θ: $\frac{d\theta}{d\Omega} = \left(\frac{ze^2}{8\pi\epsilon_0 E}\right)^2\frac{1}{\sin^4(\theta/2)}$

$N = \left(\frac{A_d nLI_0}{r_d^2}\right)\left(\frac{Ze^2}{8\pi\epsilon_0 E}\right)^2\frac{1}{\sin^4(\theta/2)}$ I_0 is beam intensity

Bohr Atom

$L = mvr = n\hbar$

Quantized radius: $r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{m_e Z e^2}$ m_e can be diff from e$^-$ mass

Quantized energy: $E_n = -\frac{m_e Z^2 e^4}{8\epsilon_0 h^2}\frac{1}{n^2}$

Ground state E for H: $E_1 = -13.6\,\text{eV}$ $n = Z = 1$ for H

E in nth orbit for H: $E_n = -\frac{E_0}{n^2} = -\frac{13.6\,\text{eV}}{n^2}$

Rydberg formula: $\frac{1}{\lambda_{mn}} = \underbrace{\frac{m_e Z^2 e^4}{8\epsilon_0^2 h^3 c}}_{R_H}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$ $m < n$

$R_H = 1.097 \times 10^7\,\text{m}^{-1}$ $m = 1$ is ground state

To $m = 1$: Lyman (UV), $m = 2$: Balmer (vis), $m = 3$: Paschen (IR)

Wave-Particle Duality

Photon Momentum: $p = \frac{h}{\lambda}$ $\quad \mathbf{p}_\gamma = \hbar \mathbf{k} \quad |\mathbf{k}| = \frac{2\pi}{\lambda}$

de Broglie wavelenth: $\lambda = \frac{h}{p}$ $\quad E = \hbar\omega$

Non-relativistic: $\lambda = \frac{h}{\sqrt{2mE_k}}$ $\quad E_k = \frac{3}{2}kT$

Relativistic: $\lambda = \frac{\lambda_c}{\sqrt{2(E_k/E_0) + (E_k/E_0)^2}}$ $\quad \lambda_c = \frac{h}{mc}$

Ultra-relativistic ($E_k \gg E_0,\ E \approx pc$): $\lambda = \lambda_c \frac{E_0}{E_k} = \frac{hc}{E_k}$

Wave Packets

$\psi_{xx} = \frac{1}{c^2}\psi_{tt} \qquad \psi = \psi_1 + \psi_2 = 2Ae^{i\bar{k}x}e^{-i\bar{\omega}t}\cos\left(\frac{1}{2}\Delta kx - \frac{1}{2}\Delta\omega t\right)$

$\bar{k} = \frac{k_1 + k_2}{2},\ \bar{\omega} = \frac{\omega_1 + \omega_2}{2}$

Phase velocity: $v_p = \frac{\omega}{k}$ $\qquad$ Group velocity: $v_g = \frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk}$

Uncertainty relations: $\Delta k \Delta x \approx 1,\ \Delta\omega\Delta t \approx 1$

Heisenberg Uncertainty Principle

$\psi(x,t) = \int_0^\infty \int_0^\infty A(k,\omega)e^{i(kx-\omega t)}\ dk\ d\omega$ (2D Fourier transform)

$\sigma_x \sigma_k = \frac{1}{2},\ \Delta x = \sigma_x,\ \Delta p = \hbar\sigma_k \qquad \Delta p \Delta x \ge \frac{\hbar}{2} \qquad \Delta E \Delta t \ge \frac{\hbar}{2}$

Particle in a Box of Length L

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \implies \Delta p \ge \frac{\hbar}{L}$

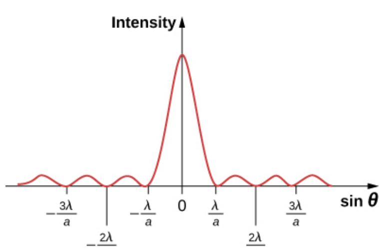
$E_k = \frac{\langle p^2 \rangle}{2m},\ \langle p \rangle = 0 \implies \langle p^2 \rangle = (\Delta p)^2 \implies \langle E_k \rangle \ge \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2mL}$

Single Slit Diffraction

$\psi(\theta) = A\cos(kr - \omega t)\ \text{sinc}\left(\frac{\pi}{\lambda}a\sin\theta\right) \qquad I = I_0\ \text{sinc}^2\left(\frac{\pi}{\lambda}a\sin\theta\right)$

At min: $\Delta L = a\sin\theta = n\lambda \qquad n = \pm1, \pm2, ...$ (NOT 0)

$\theta \approx \frac{\Delta p_y}{p_x},\ \theta \ll 1 \qquad \Delta p_y \ge \frac{\hbar}{2\Delta x}, \quad \Delta x = \frac{a}{2}, \quad p_x = \frac{h}{\lambda}$



Double Slit Diffraction

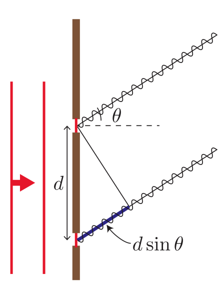
Narrow Slit

$\psi(\theta) = 2A\cos\left(\frac{\pi d}{\lambda}\sin\theta\right)\cos(kr - \omega t)$

$I = I_0\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$

For $m = 0, \pm1, \pm2, ...$

- Max I: $d\sin\theta = m\lambda$
- Min I: $d\sin\theta = \left(m + \frac{1}{2}\right)\lambda$



Schrödinger Equation

$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x,t) \qquad \frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi$

$\psi(x,t) = Ae^{i(kx-\omega t)} \qquad \int_{-\infty}^{\infty}|\psi|^2\ dx = 1, \quad |\psi|^2 = \psi^*\psi$

$\psi(x,t) = \Psi(x)\phi(t)$

$\implies \phi(t) = e^{-iEt/\hbar}$

$\implies -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi(x) = E\Psi(x)$ (time-indep Schrö eq)

Infinite Square Well

$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi(x) \implies \frac{d^2\Psi}{dx^2} = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\Psi(x)$

$\Psi(0) = \Psi(L) = 0, \quad V(x) = 0,\ 0 < x < L, \quad \int_0^L |\psi(x)|^2\ dx = 1$

$\implies \Psi(x) = \sqrt{\frac{2}{L}}\sin(kx) \implies \Psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right),\ n \in \mathbb{Z}$

$\implies k = \frac{n\pi}{L} \implies p = \hbar k = \frac{n\hbar\pi}{L} \implies E_n = \frac{p_n^2}{2m} = \frac{n^2\hbar^2\pi^2}{2mL^2}$

$\psi(x,t) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)e^{-i\omega_n t} = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)e^{-iE_n t/\hbar}$

Finite Potential Well

Inside well: $-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi(x)$

$\Psi(x) = A_1e^{ikx} + A_2e^{-ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar}$

Outside well: $-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = (E - V_0)\Psi(x)$

$\Psi(x) = B_1e^{\alpha x} + B_2e^{-\alpha x}, \quad \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

Well of length $x \in [-a, a]$ with potential V_0:

$B_1 = 0$ for $x > a$ and $B_2 = 0$ for $x < -a$

At $x = a$, $A_1e^{ika} + A_2e^{-ika} = B_2e^{-\alpha a}$

At $x = -a$, $A_1e^{-ika} + A_2e^{ika} = B_1e^{-\alpha a}$

$\implies \tan ka = \frac{\alpha}{k},\ -\cot ka = \frac{\alpha}{k}$

Quantum Mechanics

Expectation: $\langle x \rangle = \int_{-\infty}^{\infty}\psi^*(x,t)x\psi(x,t)\ dx$

Momentum operator: $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad \langle p \rangle = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi\ dx$

Hamiltonian operator: $\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + \hat{V}$

$\implies \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \implies \hat{H}\Psi(x) = E\Psi(x)$

Quantum Simple Harmonic Oscillator

$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}\underbrace{m\omega^2}_{k}x^2 \qquad \hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$

$[A, B] = AB - BA \qquad [x, \hat{p}] = i\hbar \qquad [\hat{a}_-, \hat{a}_+] = 1$

$[A,B]=0 \implies A$ and B can be measured simultaneously.

$\hat{H}=\hbar\omega\left(\hat{a}_+\hat{a}_-+\frac{1}{2}\right)=\hbar\omega\left(\hat{a}_-\hat{a}_+-\frac{1}{2}\right)$

Raising operator: $\hat{H}(\hat{a}_+\Psi)=(E+\hbar\omega)(\hat{a}+\Psi)$

Lowering operator: $\hat{H}(\hat{a}_-\Psi)=(E-\hbar\omega)(\hat{a}_-\Psi)$

Ground state: $\hat{a}_-\Psi_0(x)=0$

$E_n=\left(n+\frac{1}{2}\right)\hbar\omega$

$\Psi_n(x)=N_n e^{-\beta^2x^2/2}H_n(\beta x)$ $\qquad \beta=\sqrt{\frac{m\omega}{\hbar}}$

Physicist's Hermite polynomial: $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$

$H_0(x)=1,\ H_1(x)=2x,\ H_2(x)=4x^2-2,\ H_3(x)=8x^3-12x$

Reflection & Transmission

Step Potential of $V_0<E$

$x<0:\frac{d^2\Psi}{dx^2}=-k_1^2\psi(x),\quad k_1=\frac{\sqrt{2mE}}{\hbar}$

$x>0:\frac{d^2\Psi}{dx^2}=-k_2^2\psi(x),\quad k_2=\frac{\sqrt{2m(E-V_0)}}{\hbar}$

For $x<0:\Psi_1(x)=\underbrace{Ae^{ik_1x}}_{L\to R}+\underbrace{Be^{-ik_1x}}_{R\to L}$

For $x>0:\Psi_2(x)=Ce^{ik_2x}$

BC: $A+B=C,\quad k_1A-k_1B=k_2C$

$R=\frac{|B|^2}{|A|^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\qquad T=\frac{k_2}{k_1}\frac{|C|^2}{|A|^2}=\frac{4k_1k_2}{(k_1+k_2)^2}\qquad R+T=1$

Quantum Tunneling

Barrier potential of $V_0>E$ from $x\in[0,L]$.

$x<0:\Psi_1(x)=Ae^{ik_1x}+Be^{-ik_1x}$

$0<x<a:\Psi_2(x)=Ce^{-\alpha x}+De^{\alpha x}$

$x>a:\Psi_3(x)=Fe^{ik_1x}+Ge^{-ik_1x}$

$T(L,E)=\frac{1}{\cosh^2(\beta L)+(\gamma/2)^2\sinh^2(\beta L)}\approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta L}$

$\gamma=\frac{\beta}{k}-\frac{k}{\beta},\quad k=\frac{\sqrt{2mE}}{\hbar}\qquad \beta=\frac{\sqrt{2m(V_0-E)}}{\hbar}$

Young's Double Slit Experiment

$\frac{d^2\Psi}{dx^2}=-k^2\Psi(x)\implies\Psi(x)=Ae^{ikx}$

At screen: $\Psi(x)=A(e^{ikx}+e^{ik(x+\Delta x)})=2Ae^{ikx}e^{ik\Delta x/2}\cos\left(\frac{k\Delta x}{2}\right)$

$|\Psi(x)|^2=4|A|^2\cos^2\left(\frac{k\Delta x}{2}\right)$

Polarized Light

Consider $\rightarrow$ filter, 45° filter, and $\uparrow$ filter, where $\rightarrow\perp\uparrow$ filter.

$|\Psi_0|^2=0.5\langle\rightarrow|\rightarrow\rangle+0.5\langle\uparrow|\uparrow\rangle$

$|\Psi_1|^2=0.5\langle\rightarrow|\rightarrow\rangle\implies\Psi_1=\frac{1}{\sqrt{2}}|\rightarrow\rangle$

We're at 45° filter (think $\cos\theta|\rightarrow\rangle+\sin\theta|\uparrow\rangle$):

$|45°\rangle=\frac{1}{\sqrt{2}}(|\rightarrow\rangle+|\uparrow\rangle)\qquad|-45°\rangle=\frac{1}{\sqrt{2}}(|\rightarrow\rangle-|\uparrow\rangle)$

$\implies|\rightarrow\rangle=\frac{1}{\sqrt{2}}(|45°\rangle+|-45°\rangle)\qquad|\uparrow\rangle=\frac{1}{\sqrt{2}}(|45°\rangle-|-45°\rangle)$

$\implies\Psi_1=\frac{1}{\sqrt{2}}|\rightarrow\rangle=\frac{1}{2}(|45°\rangle+|-45°\rangle)\implies\Psi_2=\frac{1}{2}|45°\rangle$

$\Psi_2=\frac{1}{2\sqrt{2}}(|\rightarrow\rangle+|\uparrow\rangle)$

$\Psi_3=\frac{1}{2\sqrt{2}}|\uparrow\rangle\implies|\Psi_3|^2=\frac{1}{8}\langle\uparrow|\uparrow\rangle=\frac{|\Psi_0|^2}{8}$

Atomic Physics

3D Schödinger eq: $-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t)+V(\mathbf{r},t)\Psi(\mathbf{r},t)=i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$

Infinite Square Well

$\Phi(\mathbf{r})=\sqrt{\frac{8}{L_xL_yL_z}}\sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)$

$E_{n_xn_yn_z}=\frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)\qquad\phi(t)=e^{-iEt/\hbar}$

Ground state when $n_x=n_y=n_z=1$. If 2 or more dimensions of box equal, then the excited states are degenerate.

If $(L_x>L_y)\wedge(L_x>L_Z)$, first excited state is $(n_x,n_y,n_z)=(2,1,1)$.

Schrödinger Equation in Spherical Coordinates for Hydrogen Atom

Separation of variables: $\Psi(r,\theta,\phi)=R(r)Y(\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$

$Y=Y(\theta,\phi)$ is spherically harmonic

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}[E-V(r)]=-\frac{1}{\sin\theta}\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}\right)\right]=\underbrace{l(l+1)}_{\text{const}}$$

$$-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-l(l+1)\sin^2\theta=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\underbrace{-m^2}_{\text{const}}$$

$\Phi(\phi)=e^{im\phi},\quad m\in\mathbb{Z}$

$\Theta(\theta)=N_{lm}P_m^l(\cos\theta),\quad P_m^l(x)=(1-x^2)^{m/2}\frac{d^lP_m(x)}{dx^l}$

$\implies l\in\mathbb{Z},\ |m|\le l$

$\hat{E}_k\Psi+V\Psi=E\Psi\implies\left(\frac{\hat{p}_r^2}{2m}+\frac{\hat{L}^2}{2mr^2}\right)\Psi+V\Psi=E\Psi$

$\hat{p}_r^2=-\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

$\hat{L}^2=-\hbar^2\frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial^2}{\partial\phi^2}\right]$

$\implies\hat{L}^2Y_{lm}(\theta,\phi)=l(l+1)\hbar^2Y_{lm}(\theta,\phi)$

$\implies\hat{L}_z^2Y(\theta,\phi)=m^2\hbar^2Y(\theta,\phi)$

$|\mathbf{L}|=\sqrt{l(l+1)}\hbar,\ l\in\mathbb{N}\qquad$ (L quantized)

$L_z=m\hbar=\sqrt{l(l+1)}\hbar\cos\theta\qquad m=0,\pm1,\dots,\pm l$

$\theta=\cos^{-1}\left(\frac{m}{\sqrt{l(l+1)}}\right)\qquad\theta_{\min}$ if $m=l$; $2l+1$ values for L_z

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\left[-\frac{e^2}{4\pi\epsilon_0 r}+\frac{\hbar^2 l(l+1)}{2mr^2}\right]R(r)=ER(r)$$

$$\implies E_n=-\frac{m_e}{2}\left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2\frac{1}{n^2}=-\frac{E_0}{n^2} \qquad E_0=13.6\,\text{eV}$$

Quantum Numbers

- n: principal quantum number (QN); dets E lvl of e^-; $n\in\mathbb{N}\setminus\{0\}$
- l: angular momentum QN; gives $|\mathbf{L}|$; $l=0,1,\ldots,(n-1)$
  — $l=0:\ s,\quad l=1:\ p,\quad l=2:\ d,\quad l=3:\ f$
- m: angular momentum projection QN; $m=0,\pm1,\ldots,\pm l$
  — aka magnetic QN (m_l)
- s: spin QN; 1/2 for fermions
- m_s: spin projection QN; $m_s=\pm1/2$

For $l=0$: $P(r)\,dr=|\psi_{n00}|^2 4\pi r^2\,dr$

Orbital Angular Momentum

$\mu=IA=\frac{1}{2}qvr \qquad I=\frac{q}{T}=\frac{qv}{2\pi r} \qquad A=\pi r^2$

$\mu=\frac{e}{2m_e}L=\mu_B\sqrt{l(l+1)} \qquad \mu_B=\frac{e\hbar}{2m_e}=9.27\times10^{-24}\,\text{J/T}$

$\boldsymbol{\mu}=-\frac{e}{2m_e}\mathbf{L}$

Stern-Gerlach Expt & Spin-Orbit Coupling

$\boldsymbol{\mu}=-\frac{g_L\mu_B}{\hbar}\mathbf{L}=\frac{e}{m_e}\mathbf{S} \qquad \mu_s=\frac{e}{2m_e}S$

$\mathbf{F}=-\nabla U=-\nabla(-\mathbf{u}\cdot\mathbf{B})\implies F_z=\mu_z\frac{dB}{dz}=-\mu_B m_l g_L\frac{dB}{dz}$

- Expected 1 line on screen but saw 2 lines.
- 1 line = uniform, constant field.
- 2 lines = non-uniform field.

Spin angular momentum: $|\mathbf{S}|=\sqrt{s(s+1)}\hbar \qquad \mu_z=-g\mu_B m_s$

$S_z=m_s\hbar,\ m_s=\pm\frac{1}{2}$ for e$^-$ $\qquad \mu_z=-\frac{e}{2m_e}=\pm\mu_B\hbar$

Zeeman Effect

$U=-\mu_z B=-(\underbrace{-\mu_B m_l}_{\mu_z})B=m_l\mu_B B \qquad \mu_z=\mu\cos\theta$

Splitting of energy lvls by B_{ext}. Larger B means larger splitting.

# of splits $=2l+1$ E.g., $l=1\to l=0$ has 3 lines.

Pauli exclusion principle: no two e$^-$ can be in exactly the same physical state at the exact same time.

Total Angular Momentum

$\mathbf{J}=\mathbf{L}+\mathbf{S} \qquad j=l+s$ or $j=l+m_s$

$J=\sqrt{j(j+1)}\hbar \qquad J_z=m_j\hbar \qquad$ # states of $m=2j+1$

Nuclear Physics

Hyperfine Splitting

$\mathbf{F}=\mathbf{I}+\mathbf{J} \qquad \mathbf{I}$: nuclear ang mom, $\quad\mathbf{J}$: e$^-$ ang mom

$F=\sqrt{f(f+1)}\hbar,\quad f=|i-j|,\ldots,i+j$

$\Delta E=g_M m_l\mu_N B_e \qquad \mu_n=\frac{e\hbar}{2m_p}$, nuclear magneton

$\alpha:\ {}^A_Z X\to{}^{A-4}_{Z-2}Y+{}^4_2\alpha$ (due to quantum tunelling)

β decay due to weak force

e^- emission: ${}^A_Z X\to{}^A_{Z+1}Y+{}^0_{-1}e^-+\overline{\nu}_e$

p^+ emission: ${}^A_Z X\to{}^A_{Z-1}Y+{}^0_1 e^++\nu_e$

e^- capture: ${}^A_Z X+{}^0_{-1}e^-\to{}^A_{Z-1}Y+\nu_e$

$\gamma:\ {}^A_Z X^*\to{}^A_Z X+\gamma$

Atomic radius: $R=R_0A^{1/3},\quad R_0=1.2\,\text{fm},\quad A\equiv\#$ of nucleons

Binding Energy and Radioactivity

$E_b=\underbrace{(Zm_p+Nm_n-M_A)}_{\Delta m}c^2, \qquad BEN=\frac{E_b}{A}, \qquad A$ is atomic #

Fe-56 is most stable atomic nucleus, stable $N=\{20,28,50,82,126\}$

$A<56$ releases E when extra nucleon added, *fusion* more likely

$A>56$ requires E when extra nucleon added, *fission* more likely

$\frac{dN}{dt}=-\lambda N\implies N=N_0e^{-\lambda t},\ N=N_0(2)^{-t/t_{1/2}} \qquad t_{1/2}=\frac{\ln2}{\lambda}$

For decay series where $b\to a$: $\frac{dN_a}{dt}=-\lambda_aN_a+\lambda_bN_b$

Strong Force

At short range, Coulomb potential dominates: $U_c\propto1/r$

At $O(1\times10^{-15}\,\text{m})$, strong force dominates: $U_s\propto-1/r^m,\ m>1$

Force transmitted via π meson

At short range, repulsive U dominates: $U_{\text{exclusion}}\propto1/r^n,\ n>m$

As radius ↓, E lvls ↑ spacing (think Pauli exclusion)

Quantum Mechanics & Relativity

Klein-Gordon Equation

$$\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi+m^2\phi=0$$

4-vector gradient: $\partial_\mu=\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right),\quad\partial^\mu=\partial_\mu^T$

$(\partial_\mu\partial^\mu+m^2)\phi(\mathbf{x},t)=0\equiv(\Box^2+m^2)\phi(\mathbf{x},t) \qquad \Box^2$ is d'Alembertian

Plane wave soln: $\phi(\mathbf{x},t)=Ce^{-iEt+i\mathbf{px}}\implies(-E+p^2+m^2)\phi=0\implies E=\pm\sqrt{p^2+m^2}$ issue of negative E

Dirac Equation

$i\frac{\partial\psi}{\partial t}=(-i\boldsymbol{\alpha}\cdot\nabla\beta m)\psi(\mathbf{x},t)$ or $E\psi(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\psi$

$\implies-\frac{\partial^2\psi}{\partial t^2}=(-\nabla^2+m^2)\psi(x)$ or $E^2\psi=(\mathbf{p}^2+m^2)\psi$

- ψ is now a 4-component vector called a spinor.
- Solutons to Dirac eq have positive definite probability density.
- Dirac: spin-1/2 particles; Klein-Gordon: spin-0 particles.
- Dirac says E lvls symmetric about $E=0$. To prevent +ve E e$^-$ going to -ve E states, all -ve E states are filled (Pauli exclusion) $\implies$ vaccum is sea of $E<0$ e$^-$.
- Anti-particles due to absence of e$^-$ w/ $E<0$ (i.e., a hole).

Feynman interpretation: -ve E particles propagating back in time $\equiv$ +ve E particles propagating forward in time.