PHYS 244 Formula Sheet

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Newtonian Mechanics in Polar Coordinates

$\hat{r} = \langle \cos\phi, \sin\phi \rangle \qquad \hat{\phi} = \langle -\sin\phi, \cos\phi \rangle \qquad \frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi} \qquad \frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r}$

$\mathbf{r} = r\hat{r} \qquad \mathbf{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \qquad \mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$

$\dot{\phi} = \omega \qquad \ddot{\phi} = \alpha \qquad v = \omega r \qquad a = \alpha r$

$$\begin{cases} \sum F_r = m(\ddot{r} - r\dot{\phi}^2) \\ \sum F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases} \qquad \begin{cases} \text{Centrifugal term:} & -r\dot{\phi}^2 \\ \text{Coriolis term:} & 2\dot{r}\dot{\phi} \end{cases}$$

Newtonian Mechanics in Cylindrical Coordinates

$\sum F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) \qquad \sum F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \qquad \sum F_z = m\ddot{z}$

Newtonian Mechanics in Intrinsic Coordinates

$\mathbf{v} = v\hat{u}_t \qquad \mathbf{a} = \dot{v}\hat{u}_t + \frac{v^2}{r}\hat{u}_n \equiv a_t\hat{u}_t + a_n\hat{u}_n$

$\sum F_t = ma_t \qquad \sum F_n = ma_n = \frac{mv^2}{\rho} \qquad \sum F_b = 0$

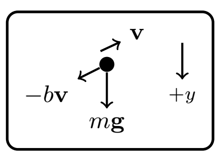
$\rho(x) = \frac{(1 + [f'(x)]^2)^{3/2}}{|f''(x)|} \qquad \rho(t) = \frac{|r'(t)|^3}{|r'(t) \times r''(t)|} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}$

Linear Air Resistance

$m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v}$

$v_x(t) = v_{0x}e^{-bt/m}$

$v_y(t) = v_{0y}e^{-bt/m} + v_{T,y}(1 - e^{-bt/m})$



$x(t) = x_0 + v_{0x}\tau(1 - e^{-t/\tau})$

$y(t) = y_0 + (v_{0y} - v_{T,y})\tau(1 - e^{-t/\tau}) + v_{T,y}t \qquad \tau = \frac{m}{b}$

$y(x) = \frac{v_{0y} + v_{T,y}}{v_{0x}}x + v_{T,y}\tau \ln\left(1 - \frac{x}{v_{0x}\tau}\right) \qquad v_{T,y} = \frac{mg}{b}$

If $+y$ is up, then flip sign on $v_{T,y}$.

Quadratic Air Resistance

$m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v}^2 \qquad v_T = \sqrt{\frac{mg}{b}}$

x-dir only: $v_x(t) = \frac{v_{0x}}{1 + cv_{0x}t/m} \qquad x(t) = \frac{mv_{0x}}{c}\ln\left(1 + \frac{cv_{0x}t}{m}\right)$

y-dir only: $v_y(t) = v_T \tanh\left(\frac{2gt}{v_T}\right) \qquad y(t) = y_0 + \frac{v_T^2}{g}\ln\left(\cosh\frac{gt}{v_T}\right)$

Charged Particle in Uniform Magnetic Field

$\mathbf{F_B} = q\mathbf{v} \times \mathbf{B} = \omega\langle v_y, -v_x \rangle \qquad$ Cyclotron freq: $\omega = \frac{qB}{m}$

$\eta(t) = \eta_0 e^{-i\omega t} \qquad \eta(t) = v_x(t) + iv_y(t)$

$v_x(t) = v_{0x}\cos\omega t + v_{0y}\sin\omega t \qquad v_y(t) = v_{0y}\cos\omega t - v_{0x}\sin\omega t$

$x(t) = x_0 + \frac{v_{0x}}{\omega}\sin\omega t + \frac{v_{0y}}{\omega}(1 - \cos\omega t)$

$y(t) = y_0 + \frac{v_{0y}}{\omega}\sin\omega t - \frac{v_{0x}}{\omega}(1 - \cos\omega t)$

$v_x(t)^2 + v_y(t)^2 = v_{0x}^2 + v_{0y}^2 = v_0^2$

$$\left[x(t) - x_0 - \frac{v_{0y}}{\omega}\right]^2 + \left[y(t) - y_0 + \frac{v_{0x}}{\omega}\right]^2 = \frac{v_{0x}^2}{\omega^2} + \frac{v_{0y}^2}{\omega^2} = R^2$$

Linear Momentum

$\Delta\mathbf{P} = \sum_i \int_0^t \mathbf{F}_{\text{ext},i}\, dt = \mathbf{J}_{\text{net}}$

$e = \frac{v_{b2n} - v_{a2n}}{v_{a1n} - v_{b1n}} \qquad$ n-axis is along line of impact, t-axis $\perp$ n-axis

Geometry known:

- $m_a v_{a1n} + m_b v_{b1n} = m_a v_{a2n} + m_b v_{b2n}$
- $v_{a1t} = v_{a2t},\ v_{b1t} = v_{b2t}$

Geometry unknown: conserve total $\mathbf{p}$ in any 2 perpendicular dirs.

Center of Mass

$\mathbf{r}_{\text{cm}} = \frac{1}{M}\sum_i m_i\mathbf{r}_i \qquad \mathbf{v}_{\text{cm}} = \dot{\mathbf{r}}_{\text{cm}} = \frac{1}{M}\sum_i m_i\mathbf{v}_i$

$\mathbf{a}_{\text{cm}} = \ddot{\mathbf{r}}_{\text{cm}} = \frac{1}{M}\sum_i m_i\mathbf{a}_i \qquad M\mathbf{a}_{\text{cm}} = \mathbf{F}_{\text{ext,sys}}$

$\mathbf{v}_{b/a} = \mathbf{v}_{b/c} - \mathbf{v}_{a/c} \qquad$ c is an arbitrary inertial frame

Rocket equation: $m\dot{v} = F_{\text{ext,sys}} - v_{\text{ex}}\dot{m}$

If $F_{\text{ext,sys}} = 0$: $v(m) = v_0 + v_{\text{ex}}\ln\frac{m_0}{m}$

Torque and Angular Momentum

$\mathbf{l} = \mathbf{r} \times \mathbf{p} \qquad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad \boldsymbol{\tau} = \frac{d\mathbf{l}}{dt}$

Central force: $\mathbf{F} = F(r)\hat{r} \qquad \tau_F = \mathbf{r} \times (F\hat{r}) = 0 \implies \mathbf{l} = \text{const}$

Kepler's 2nd law: $\dot{A} = \frac{1}{2}r^2\dot{\phi} = \frac{l}{2m} \implies A(t) = \frac{l}{2m}t$

$\frac{d\mathbf{L}_0}{dt} = \boldsymbol{\tau}_{\text{ext, sys}}, \quad \boldsymbol{\tau}_{\text{ext, sys}} = I\boldsymbol{\alpha}_z$

$L_z = I_z\omega_z, \quad I_z = \sum_i m_i r_i^2$ or $I_z = \int r^2\, dm$

Parallel axis thm: $I_p = I_{\text{cm}} + Md^2$

$T = \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}Mv_{\text{cm}}^2$

Energy

$W = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}(t) \qquad$ WET: $W_{\text{tot}} = \frac{1}{2}m(v_2^2 - v_1^2) = \Delta T$

$W_{\text{g}} = -mg\Delta y = -mg(y - y_0)$

Conservative $\mathbf{F}$: $W_F = -\Delta U = U_1 - U_2$

$\nabla \times \mathbf{F} = 0 \implies \mathbf{F}$ is conservative ($W = 0$ on closed path)

2D: $\mathbf{F} = P\hat{i} + Q\hat{j} \implies Q_x - P_y = 0 \implies \mathbf{F}$ conservative

Option 1: $\mathbf{F} = -\nabla U \qquad$ Option 2: line integral on easy path

If $W_{nc} = 0$, then $\Delta E = 0 \iff \Delta T = -\Delta U$

1D only: $E = \frac{1}{2}mv^2 + U(x)$, then

$$v = \pm\sqrt{\frac{2(E - U(x))}{m}} \implies t = \int_{x_0}^{x} \frac{\pm dx}{\sqrt{2(E - U(x))/m}}$$

$F(x) = -\frac{dU(x)}{dx}$ Constrained force: $F_s = -\frac{dU}{ds}$

If $dU/dx > 0$, PE $\uparrow$ for increasing $x \implies F < 0$ (F points to $-x$)

If $dU/dx < 0$, PE $\downarrow$ for increasing $x \implies F > 0$ (F points to $+x$)

A central force is conservative iff it's spherically symmetric.

$\mathbf{F}_{\text{central}} = f(\mathbf{r})\hat{r} = f(r, \phi, \theta)\hat{r}$

Spherically symm: $f(r, \phi, \theta) \mapsto f(r)$

$$U_{\text{int, sys}} = \sum_i \sum_{j>i} U_{ij}$$

Mech energy: $E = \sum_i \frac{1}{2} m_i v_i^2 + \sum_i \sum_{j>i} U_{ij} + \sum_i U_{i,\text{ext}}$

$W_{\text{non-cons}} = -\Delta U_{\text{int}}$ $\sum_{\text{universe}} (K + U + U_{\text{int}}) = \text{const}$

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Oscillations

$$U(x) \approx \frac{1}{2} U''(0) x^2 + \frac{1}{3!} U'''(0) x^3 + \cdots$$

$k := U''(0)$ $k > 0$ for stable eq

$\ddot{x}(t) + \omega^2 x(t) = 0$ $\omega = \sqrt{k/m}$

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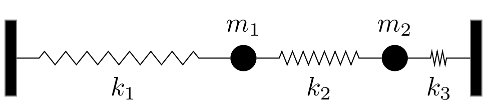
Coupled Oscillators



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\implies -\omega^2 \mathbf{M}(\mathbf{a}e^{i\omega t}) = -\mathbf{K}(\mathbf{a}e^{i\omega t}) \implies \det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

1. Find characteristic equation and solve for eigenvalues.
2. Plug eigenvalues back to solve for eigenvectors.
3. $\mathbf{z}_i(t) = A_i(\text{eigval})(\text{eigvec})e^{i(\omega_i t - \delta_i)} = C_i(\text{eigval})(\text{eigvec})e^{i\omega_i t}$.
4. $\mathbf{x}(t) = \text{Re}\{\mathbf{z}(t)\}$. Each $\mathbf{x}_i(t)$ is a normal mode.

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Case 1: $m_1 = m_2,\ k_1 = k_2 = k_3$

$\mathbf{x}_1(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_0 t - \delta)$ $\mathbf{x}_2(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{3}\omega_0 t - \delta)$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

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Case 2: $m_1 = m_2,\ k_2 \ll k_1 = k_3 = k$

$$\mathbf{x}(t) = \left(C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\epsilon t} \right) e^{i\omega_{\text{avg}} t}$$

$\omega_1 = \sqrt{\frac{k + 2k_2}{m}} = \omega_{\text{avg}+\epsilon}$ $\omega_2 = \sqrt{\frac{k}{m}} = \omega_{\text{avg}-\epsilon}$

$\omega_{\text{avg}} = \frac{\omega_1 + \omega_2}{2}$ $\epsilon = \frac{\omega_2 - \omega_1}{2} \ll \omega_{\text{avg}}$

If $C_1 =_2= \frac{A}{2}$, then $\mathbf{z}(t) = A \begin{bmatrix} \cos \epsilon t \\ i \sin \epsilon t \end{bmatrix} (\cos \omega_{\text{avg}} t + i \sin \omega_{\text{avg}} t)$.

$$\mathbf{x}(t) = \text{Re}\{\mathbf{z}(t)\} = A \begin{bmatrix} \cos \epsilon t \cos \omega_{\text{avg}} t \\ -\sin \epsilon t \sin \omega_{\text{avg}} t \end{bmatrix} \implies \text{beats}$$

Linear Damped SHO

$m\ddot{x} = -kx - b\dot{x} \implies \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ $2\beta = \frac{b}{m},\ \omega_0^2 = \frac{k}{m}$

$$\hat{D} = \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 \implies \hat{D}x = 0$$

$$\beta > \omega_0 \implies x(t) = C_1 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t}$$

$$\beta = \omega_0 \implies x(t) = e^{-\beta t}(C + Dt)$$

$$\beta < \omega_0 \implies x(t) = e^{-\beta t}(A \cos \omega_d t + B \sin \omega_d t),\ \omega_d = \sqrt{\omega_0^2 - \beta^2}$$

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$\hat{D}x = f_0 \cos \omega t$ $f_0 = \frac{F_{\max}}{m},\ \beta = -\frac{b}{2m},\ \omega_0 = \sqrt{\frac{k}{m}}$

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}}_{A(\omega)} \cos(\omega t - \delta)$$

$A(\omega)$ is a Lorentzian fn $\omega_{\max} = \sqrt{\omega_0^2 - 2\beta^2}$

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Lagrangian Mechanics

$L = T - U, \quad L = L(q, \dot{q}, t)$

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\ dt \implies \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

- If $F(y, y', x) = F(y', x)$, then $\frac{\partial F}{\partial y'} = \text{const}$
- Beltrami identity: $F - y' \frac{\partial F}{\partial y'} = C$

$$L = T - U,\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \iff m\ddot{x}_i = -\frac{\partial U}{\partial x_i}$$

Hamilton's principle: $\delta S = \delta \int_{t_0}^{t_f} L\ dt = 0$

$\frac{d}{dt}$(generalized momentum) = generalized force

If E-L eq holds for $x_i = x_i(q_1, q_2, \ldots, q_n; t)$, then it also holds for the q_i coordinates. That is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}, \quad 1 \leq m \leq n$$

Constraint forces are automatically incorporated with the right choice of generalized variables.

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Generalized Coordinates and Degree of Freedom

Parameters $\{q_1, \ldots, q_n\}$ called a set of generalized coordinates if $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \ldots, q_n, t)$ and $q_i = q_i(\mathbf{r}_1, \ldots, \mathbf{r}_n, t)$ s.t. n is the smallest number that allows the system to be parameterized this way.

Natural coordinates: relation btw $\mathbf{r}$ and q is time-indep

Dof: coordinate that can be varied indep of others

Holonomic sys: #dof = #generalized coordinates

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Conservation Laws

Spatial translational invariance $\iff$ cons of total momentum

Time translational invariance $\iff$ cons of E (think Beltrami)

Hamiltonian: $H = \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L$

H is the Legendre transform of L

1. If the relation btw generalized coords q_i and Cartesian coords is time-indep, then H is the E of the sys.
2. If L as no explicit time dependence, then H is conserved.

Noether's thm. For each continuous symmetry of L, there is exists a conserved quantity.

Noether charge/cons of gen p: $Q_{\text{cons}} = \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i$

$K_i(q)$ are called generators of symmetry.

Lagrange Multipliers and Constraint Forces

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + \lambda(t)\frac{\partial f}{\partial q_i} \qquad \lambda(t) \text{ is the Lagrange multiplier}$$

$f(q_1, q_2) = c$, $\quad f$ is a constraint

Two-Body Central Force Problems

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \quad M = m_1 + m_2$$

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

Reduced mass: $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

$$\mu\ddot{r} = -\frac{dU}{dr} + \frac{l_z^2}{\mu r^3} \qquad U_{\text{eff}}(r) = U(r) + \frac{l_z^2}{2\mu r^2}$$

Inverse Square Law

$$U_{\text{eff}}(r) = -\frac{\gamma}{r} + \frac{l_z^2}{2\mu r^2} \qquad F(r) = -\frac{\gamma}{r^2} \iff U(r) = -\frac{\gamma}{r}$$

$$\frac{dU_{\text{eff}}}{dt} = 0 \implies r_0 = \frac{l_z^2}{\gamma\mu} \qquad \frac{d^2U_{\text{eff}}}{dt^2} = \frac{\gamma^4\mu^3}{l_z^6} > 0,\ \gamma > 0$$

$E < 0$: particle bound $\qquad$ $E > 0$, particle unbound ($r \to \infty$)

$$U(r) = -\frac{\gamma}{r} \implies r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \qquad c = \frac{l_z^2}{\gamma\mu}, \quad \epsilon = \frac{Al_z^2}{\gamma\mu}$$

$$E = \frac{\gamma^2\mu}{2l_z^2}(\epsilon^2 - 1)$$

Equation for Orbits

$$\mu\ddot{r} = -\frac{\gamma}{r^2} + \frac{l_z^2}{\mu r^3}$$

$$\dot{r} = \sqrt{\frac{2[E - U_{\text{eff}}(r)]}{\mu}} \implies t = \int_{r_0}^{r} \frac{dr}{\sqrt{2[E - U_{\text{eff}}]/\mu}}$$

$$\phi - \phi_0 = \frac{l_z}{\mu}\int_0^t \frac{dr}{[r(t)]^2}$$

We can do better (i.e., linearize the ODE)!

$$w \equiv \frac{1}{r} \implies \frac{d^2w}{d\phi^2} = \frac{\gamma\mu}{l_z^2} - w$$

$$w(\phi) = A\cos(\phi - \phi_0) + \frac{\gamma\mu}{l_z^2} = \frac{\gamma\mu}{l_z^2}\left[1 + A\frac{l_z^2}{\gamma\mu}\cos(\phi - \phi_0)\right]$$

Let $c = \frac{l_z^2}{\gamma\mu}$ and $\epsilon = \frac{Al_z^2}{\gamma\mu}$ and choose polar axes st $\phi_0 = 0$

$$\implies w(\phi) = \frac{1 + \epsilon\cos\phi}{c} \implies r(\phi) = \frac{1}{1 + \epsilon\cos\phi}$$

Bound states ($E < 0$): $\epsilon = 0$: circle, $0 < \epsilon < 1$: ellipse

Unbound states ($E \geq 0$): $\epsilon = 1$: parabola, $\epsilon > 1$: hyperbola

$$\frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) = E \implies E = U_{\text{eff}}(r_{\min}) = -\frac{\gamma}{r_{\min}} + \frac{l_z^2}{2\mu r_{\min}^2}$$

$$r_{\min} = \frac{c}{1 + \epsilon} \implies E = \frac{\gamma^2\mu}{2l_z^2}(\epsilon^2 - 1)$$

Hamiltonian Mechanics

$$H = H(q_i, p_i) \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i p_i\dot{q}_i - L$$

$H = T + U$ if q_i are time-indep

1. Choose suitable generalized coords q_i.
2. Det T and U in terms of q_i and $\dot{q}_i$.
3. Det p_i. If sys conservative, $p_i = \partial T/\partial \dot{q}_i$.
4. Solve for $\dot{q}_i$ in terms of p and q.
5. Write down H and 1st order Hamiltonian eqs of motion.