MATH 311 Midterm Summary

Eddie Guo
October 18, 2021

1 Important Equations

Theorem 1.1. (De Moivre's theorem). $(e^{it})^n = e^{int} \implies (\cos t + i\sin t)^n = \cos nt + i \sin nt$.

Corollary 1.2. $\cos n\theta = \Re(e^{in\theta})$ and $\sin n\theta = \Im(e^{in\theta})$.

$$\begin{aligned}
e^{3i\theta} &= \cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
\Re(e^{3i\theta}) &= \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta \\
\Im(e^{3i\theta}) &= \sin 3\theta = 3\cos^2\sin\theta - \sin^3\theta = 3(1-\sin^3\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Definition 1.1. (Complex logs). $\log z = \ln|z| + i\arg z = \ln|z| + i(t+2\pi k), k \in \mathbb{Z}$. Furthermore, $z^c = e^{c\log z}$.

Theorem 1.3. (nth root). If $z^n = c$, then $z = |c|^{1/n}e^{(i\arg c)/n} = |c|^{1/n}e^{i(t+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1$.

Example 1.4. Compute $i^{1/3}$.

$$i^{1/3} = e^{\frac{1}{3}\log i} = e^{\frac{1}{3}(\ln|i| + i\arg i)} = e^{\frac{i}{3}}(\pi/2 + 2\pi k), \quad k = 0, 1, 2$$

$$\begin{cases} k=0: & e^{i\pi/6} = \frac{1}{2}(\sqrt{3}+i) \\ k=1: & e^{i5\pi/6} = \frac{1}{2}(-\sqrt{3}+i) \\ k=2: & e^{i3\pi/2} = -i \end{cases}$$

2 Trigonometric Functions

Definition 2.1. The complex trig functions are

$$\begin{aligned}
&\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \cosh z = \frac{e^z+e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2} \\
&\cosh z = \cos iz, \quad i\sinh z = \sin iz, \quad \tanh z = -i\tan iz, \quad \tanh iz = i\tan z \\
&\cos z = \cos(x+iy) = \cos x\cosh y - i\sin x\sinh y \\
&\sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y
\end{aligned}$$

Example 2.1. Solve for z given $\cos z = 2$.

$$\begin{aligned}
\frac{e^{iz}+e^{-iz}}{2} &= 2 \implies e^{2iz} - 4e^{iz} + 1 = 0 \\
e^{iz} &= \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} \\
iz &= \log(2\pm\sqrt{3}), \qquad \text{Note that } (2+\sqrt{3})(2-\sqrt{3}) = 1, \text{ so } (2-\sqrt{3}) = (2+\sqrt{3})^{-1} \\
z &= \pm\frac{1}{i}\log(2+\sqrt{3}) \\
&= \pm i\log(2+\sqrt{3}) \\
&= \pm i\left[\ln(2+\sqrt{3}) + i\arg(2+\sqrt{3})\right] \\
&= \pm i\left[\ln(2+\sqrt{3}) + i(2\pi k)\right], \quad k \in \mathbb{Z} \\
&= \pm\left[i\ln(2+\sqrt{3}) + 2\pi k\right], \quad k \in \mathbb{Z}
\end{aligned}$$

3 Linear Fractional Transformations (LFTs)

Definition 3.1. (LFTs). An LFT is defined as $w = w(z) = \frac{az+b}{cz+d}$ where det $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$.

Corollary 3.1. Let $w := 1/z$. Then w takes $A(x^2+y^2)+Bx+Cy+D=0$ in the z-plane to $D(\mu^2+\nu^2)+B\mu - C\nu + A = 0$ in the w-plane, where $w(z) = \mu + i\nu$.

Example 3.2. Given $w := 1/z$, what is $|z+1-i| = \sqrt{2}$ in the w-plane?

$$|z+1-i| = \sqrt{2} \Longrightarrow (x+1)^2 + (y-1)^2 = 2$$
$$x^2+y^2+2x-2y = 0 \Longrightarrow 2u+2v+1 = 0$$

Example 3.3. Given $w := 1/z$, what is $|z-3| = 2$ in the w-plane?

$$|z-3| = 2 \Longrightarrow (x-3)^2+y^2 = 4$$
$$x^2+y^2-6x+5 = 0 \Longrightarrow 5(u^2+v^2)-6u+1 = 0$$
$$\left(u - \frac{3}{5}\right)^2 + v^2 = \left(\frac{2}{5}\right)^2 \Longrightarrow \left|w - \frac{3}{5}\right| = \frac{2}{5}$$

This is a circle in the w-plane centered at $(-3/5, 0)$ with radius $2/5$.

Lemma 3.4. Given $T(P_1) = Q_1,\ T(P_2) = Q_2,\ T(P_3) = Q_3$, we can obtain an LFT $w = w(z)$ by solving

$$\frac{z-P_1}{z-P_3}\left(\frac{P_2-P_3}{P_2-P_1}\right) = \frac{w-Q_1}{w-Q_3}\left(\frac{Q_2-Q_3}{Q_2-Q_1}\right)$$

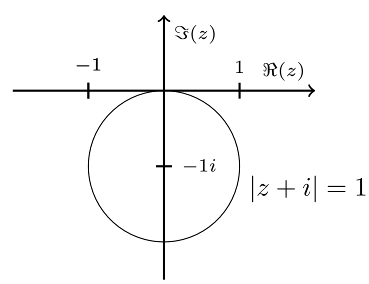
Lemma 3.5. Given $P_1,\ P_2,\ P_3$ on a circle in $\mathbb{C}$, if $T(P_1) = 0,\ T(P_2) = 1,\ T(P_3) = \infty$, then

$$T(z) = \frac{z-P_1}{z-P_3}\left(\frac{P_2-P_3}{P_2-P_1}\right)$$

maps the circle to the real axis.

Theorem 3.6. Compositions of LFTs are LFTs.

Example 3.7. Consider the unit circle $\mathcal{C} = |z+i| = 1$. Find an LFT which maps $\mathcal{C}$ to the line l given by the parametric equation $l(t) = (1+2i) + t(-2+i), t \in \mathbb{R}$. Describe the region $w(|z+i| < 1)$.



Choose 3 points on $\mathcal{C}$: $0,\ -2i,\ -1-i$. Set $T(0) = 0$, $T(-2i) = 1$, $T(-1-i) = \infty$. Then the LFT is

$$T(z) = \frac{z-0}{z-(-1-i)}\left(\frac{-2i-(-1-i)}{-2i-0}\right) = \frac{z}{z+1+i}\left(\frac{1-i}{-2i}\right)$$

Since $T(\mathcal{C})$ is the real axis, $l(z) = (-2+i)\ T(z) + (1+2i)$.

In (x, y) coordinates, $l(t)$ passes through $(1, 2)$ with slope $-1/2$: $\frac{y-2}{x-1} = -\frac{1}{2} \Longrightarrow y = y(x) = -\frac{1}{2}x + \frac{5}{2}$. Now choose a point within $\mathcal{C}$, say $-i$.

$$T(-i) = \frac{-i}{2i+1}\left(\frac{1-i}{-2i}\right) = \frac{1-i}{4i+2} = -\frac{1}{10} - i\frac{3}{10}$$
$$w(-i) = (-2+i)\ T(-i) + (1+2i) = (-2+i)\left(-\frac{1}{10} - i\frac{3}{10}\right) + (1+2i) = \frac{3}{2} + i\frac{5}{2}$$
$$y(3/2) = -\frac{1}{2}\left(\frac{3}{2}\right) + \frac{5}{2} = \frac{7}{4} < \frac{5}{2}$$

Since $w(i)$ is above $l(t)$, $w(|z+i| < 1) = \left\{z \in \mathbb{C} \mid \Im(z) > -\frac{1}{2}\Re(z) + \frac{5}{2}\right\}$. To find an LFT that maps l to $\mathcal{C}$ (i.e., the inverse map), solve for z in $w = (-2+i)\ T(z) + (1+2i)$.

4 Complex Differentiation

Definition 4.1. (Complex derivatives). The complex derivative is defined as $f'(z) = \mu_x + i\nu_x = \nu_y - i\mu_y$. The derivative exists $\iff$ the Cauchy-Riemann equations exist: $\mu_x = \nu_y, \ \mu_y = -\nu_x$.

Definition 4.2. (Domains). Domains are open, connected sets. Let $D \in \mathbb{C}$ be a domain and $p \in D$ be a point. Then,

1. f is analytic at p if $f'(z)$ exists at p AND in an epsilon neighbourhood around any p.
2. f is analytic $\forall p \in D \iff f'(z)$ is analytic $\forall z \in D$.
3. f is entire if f is analytic on $\mathbb{C}$.

Example 4.1. Let $f(z) = (xy^2 + y) + iyx^2$. Find all points $p \in \mathbb{C}$ where $f'(p)$ exists and compute $f'(p)$. Is f analytic at any p?

Let $\mu = xy^2$ and $\nu = yx^2$. By the CR equations,

$$\begin{aligned} \mu_x &= y^2 = x^2 = \nu_y \implies x = \pm y \\ \mu_y &= 2xy + 1 = -2xy = -\nu_x \implies xy = -1/4 \end{aligned}$$

Therefore, $x = \pm 1/2$ and $y = \mp 1/2$, which means $p = \left\{ \frac{1}{2}(1-i), \ \frac{1}{2}(-1+i) \right\}$.

$$f'(p) = \mu_x(p) + i\nu_x(p) = y^2 + i(2xy)|_p = \frac{1}{4} - \frac{i}{2}$$

f is nowhere analytic as f' does not exist on any neighbourhood around p.

Example 4.2. Find all points where $f(z) = \frac{\cot z}{z^4+16}$ is analytic.

$$f(z) = \frac{\cos z}{\sin z(z^4 + 16)}$$

Since $\cos z$, $\sin z$, and $z^4 + 16$ are entire, f is analytic except when the denominator is 0.

$$\begin{aligned} \sin z = 0 &\implies z = k\pi, \quad k \in \mathbb{Z} \\ z^4 + 16 = 0 &\implies z = (-16)^{1/4} = 2e^{i(2m+1)\pi/4}, \quad m = 0, 1, 2, 3 \end{aligned}$$

Thus, f is analytic where $\{z \in \mathbb{C}\} \setminus \{k\pi, \ 2e^{i\pi/4}, \ 2e^{i3\pi/4}, \ 2e^{5\pi/4}, \ 2e^{i7\pi/4}\}, \ k \in \mathbb{Z}$.

Example 4.3. Let $f(z) = \mu + i\nu$ be an entire function satisfying $A\mu + B\nu + C = 0$, where $z \in \mathbb{C}$, $A, B, C \in \mathbb{R}$, and $(A, B) \neq (0, 0)$. Show that $f \in \mathbb{C}$ is a constant function.

$$\left.\begin{aligned} A\mu_x + B\nu_x = 0 \\ A\mu_y + B\nu_y = 0 \end{aligned}\right\} \implies \begin{bmatrix} \mu_x & \nu_x \\ \mu_y & \nu_y \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \det \begin{bmatrix} \mu_x & \nu_x \\ \mu_y & \nu_y \end{bmatrix} = \det \begin{bmatrix} \mu_x & -\mu_y \\ \mu_y & \mu_x \end{bmatrix} = \mu_x^2 + \mu_y^2 = \nu_y^2 + \nu_x^2 = 0$$

By the CR equations. $\nabla \mu = \nabla \nu = 0$ implies $\mu, \nu \in \mathbb{R}$. Therefore, $f(z) = \mu + i\nu$ is a constant function.

Example 4.4. Let $f(z) = \mu + i\nu$ be an entire function satisfying $\nu = \mu^2$. Show that $f(z) \in \mathbb{C}$ is a constant function.

$$\left.\begin{aligned} \nu_x = 2\mu\mu_x \implies \nu_x - 2\mu\mu_x = 0 \\ \nu_y = 2\mu\mu_y \implies \nu_y - 2\mu\mu_y = 0 \end{aligned}\right\} \implies \begin{bmatrix} \nu_x & \mu_x \\ \nu_y & \mu_y \end{bmatrix} \begin{bmatrix} 1 \\ -2\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} \mu_x & \nu_x \\ \mu_y & \nu_y \end{bmatrix} = \det \begin{bmatrix} \mu_x & -\mu_y \\ \mu_y & \mu_x \end{bmatrix} = \mu_x^2 + \mu_y^2 = \nu_y^2 + \nu_x^2 = 0$$

By the CR equations. $\nabla \mu = \nabla \nu = 0$ implies $\mu, \nu \in \mathbb{R}$. Therefore, $f(z) = \mu + i\nu$ is a constant function.

5 Important Examples

Example 5.1. $i^{72} - 3i^{81} + 5 = (-1)^{36} - 3(-1)^{40}i + 5 = 6 - 3i$

Example 5.2. $(1-\sqrt{3}i)^{-5} = (2e^{-i\pi/3})^{-5} = 2^{-5}e^{i5\pi/3} = 2^{-5}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

Example 5.3. $\left|\frac{(3+i)^{20}}{(2-i)^3}\right| = \frac{|3+i|^{20}}{|2-i|^3} = \frac{10^{20/2}}{5^{3/2}} = \frac{10^{10}}{5\sqrt{5}}$

Example 5.4. Let the roots of the equation $(z+1)^5 + z^5 = 0$ be $z_k,\ k = 0, 1, \ldots, 4$. Show that $\Re(z_k) = -1/2$.

$$
\begin{aligned}
(z+1)^5 + z^5 = 0 \Longrightarrow \left(-\frac{z+1}{z}\right)^5 &= \left(-1 - \frac{1}{z}\right)^5 = 1 \\
-\left(1 + \frac{1}{z}\right) = e^{i(2k\pi/5)}, \quad k = 0, 1, \ldots, 4 &\Longrightarrow z_k = -\frac{1}{1+e^{i\lambda k}}, \quad \lambda = 2\pi/5 \\
&= -\frac{1}{1+\cos\lambda k + i\sin\lambda k} = -\frac{(1+\cos\lambda k) - i\sin\lambda k}{2(1+\cos\lambda k)} = -\frac{1}{2} + \frac{i\sin\lambda k}{2(1+\cos\lambda k)}
\end{aligned}
$$

Therefore, $\Re(z_k) = -1/2$ for $k = 0, 1, \ldots, 4$, which lies on the line $x = -1/2$.

Example 5.5. Express the complex number $z = \cos(\pi(1+i))$ in the form $a + ib$, where $a, b \in \mathbb{R}$.

$$
\begin{aligned}
\cos(\pi(1+i)) &= \cos(\pi + i\pi) = \cos\pi\cos i\pi - \sin\pi\sin i\pi \\
&= \cos\pi\cosh\pi - i\sin\pi\sinh\pi \\
&= -\cosh\pi \\
&= -\frac{e^{\pi} + e^{-\pi}}{2}
\end{aligned}
$$

Example 5.6. $|z|^i = |e^{i\log 2}| = |e^{i(\ln 2 + i\arg 2)}| = |e^{i\ln 2}e^{-2\pi k}| = e^{-2\pi k}, \quad k \in \mathbb{Z}$

Example 5.7. Compute $\Re\left[(1-i\sqrt{3})^{3i}\right]$.

$$
\begin{aligned}
(1-i\sqrt{3})^{3i} &= e^{3i\log(1-i\sqrt{3})} = e^{3i\left[\ln 2 + i\left(-\frac{\pi}{3} + 2\pi k\right)\right]} = e^{3i\ln 2}e^{-3\left(-\frac{\pi}{3}+2\pi k\right)}, \quad k \in \mathbb{Z} \\
&= [\cos(3\ln 2) + i\sin(3\ln 2)]e^{\pi - 6\pi k}, \quad k \in \mathbb{Z} \\
\Re\left[(1-i\sqrt{3})^{3i}\right] &= e^{(6k+1)\pi}\cos(3\ln 2), \quad k \in \mathbb{Z}
\end{aligned}
$$

Example 5.8. Find all values of $\log e^z$.

$$
\begin{aligned}
\log e^z &= \ln|e^z| + i\arg e^z \\
|e^z| &= |e^x e^{iy}| = e^x, \quad \arg e^z = \arg(e^x e^{iy}) = y + 2\pi k, \quad k \in \mathbb{Z} \\
\log e^z &= \ln e^x + i(y + 2\pi k) = (x + iy) + 2\pi k i = z + 2\pi i k, \quad k \in \mathbb{Z}
\end{aligned}
$$

Example 5.9. Show that $\log z^{1/N} = \frac{1}{N}\log z$.

$$
\begin{aligned}
z^{1/N} &= e^{\log z/N} = e^{\frac{1}{N}(\ln|z| + i\arg z)} = |z|^{1/N}e^{\frac{i}{N}\arg z} \\
\log z^{1/N} &= \ln|z|^{1/N} + \frac{i}{N}\arg z + 2n\pi i, \quad n \in \mathbb{Z} \\
&= \frac{1}{N}\ln|z| + \frac{i}{N}\arg z \\
&= \frac{1}{N}\log z
\end{aligned}
$$

Example 5.10. Show that $\log z^N \neq N\log z$.

$$
\begin{aligned}
z^N &= e^{N\log z} = e^{N(\ln|z| + i\arg z)} = |z|^N e^{iN\arg z} = |z|^N e^{iN(t + 2\pi n)}, \quad n \in \mathbb{Z} \\
\log z^N &= N\ln|z| + iNt + 2\pi n i, \quad n \in \mathbb{Z} \\
&\neq N\ln|z| + iNt + 2nN\pi i = N\log z
\end{aligned}
$$

Example 5.11. Find all values of $\log[(1+i)^3]$. Are the values the same as that of $3\log(1+i)$?

$$\log[(1+i)^3] = 3\ln\sqrt{2} + i\arg[(1+i)^3] = 3\ln\sqrt{2} + i\left(\frac{3\pi}{4} + 2\pi n\right), \quad n \in \mathbb{Z}$$
$$3\log(1+i) = 3[\ln\sqrt{2} + i\arg(1+i)] = 3\ln\sqrt{2} + 3i\left(\frac{\pi}{4} + 2\pi m\right) = 3\ln\sqrt{2} + i\left(\frac{3\pi}{4} + 6\pi m\right), \quad m \in \mathbb{Z}$$

Thus, the values are not the same.

Example 5.12. Given $w(z) = \frac{iz+1}{z+i}$.

(i) Verify that $w(z)$ is an LFT.

$\det \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} = -2 \neq 0$

(ii) Describe the image $w(|z| \leq 1)$ in the w-plane.

Choose 3 points on the circle $|z| = 1$. Since $w(i) = 0,\ w(-i) = \infty,\ w(1) = 1$, the LFT maps $|z| = 1$ to the real axis, $\Im(z) = 0$. Choose a point inside the circle: $w(0) = -i$. Therefore, $w(|z| \leq 1) = \{z \in \mathbb{C} \mid \Im(z) \leq 0\}$.

(iii) Find an LFT $w = w(z)$ that takes the real axis to the unit circle $|z| = 1$.

This is the inverse transformation, where we solve for z in terms of w:

$$w(z+i) = iz + 1 \Longrightarrow wz - iz = 1 - iw \Longrightarrow z(w-i) = 1 - iw \Longrightarrow z = \frac{1-iw}{w-1} \Longrightarrow w = \frac{-iz+1}{z-i}$$

Example 5.13. Consider the map $w = 1/z$. Describe the image $w(|z+1-2i|) = \sqrt{6}$.

$$\begin{aligned} |z+1-2i| = \sqrt{6} \iff (x+1)^2 + (y-2)^2 = 6 \iff (x^2+y^2) + 2x - 4y - 1 = 0 \\ (-1)(\mu^2+\nu^2) + 2\mu + 4\nu + 1 = 0 \\ \mu^2 + \nu^2 - 2\mu - 4\nu - 1 = 0 \\ (\mu-1)^2 + (\nu-2)^2 = 6 \\ |w - 1 - 2i| = \sqrt{6} \end{aligned}$$

This is a circle in the w-plane centered at $(1, 2i)$ with radius $\sqrt{6}$.

Example 5.14. Let $f(z) = 2x^2 - y^2 + i2xy$.

(i) Find where the Cauchy-Riemann equations hold.

$$\mu_x = 4x = 2x = \nu_y \Longrightarrow x = 0$$
$$\mu_y = -2y = 2y = -\nu_x$$

Therefore, CR equations hold when $x = 0 \Longrightarrow \Re(z) = 0$.

(ii) Find where f is differentiable.

The partial derivatives exist and are continuous wherever the CR equations hold. Thus, f' exists and is differentiable when $x = 0$, which implies $\Re(z) = 0$.

(iii) Find where the function is analytic.

f is nowhere analytic because there is no neighbourhood around any point on $\Re(z) = 0$ where f' exists.

Example 5.15. Find the image of the region $-\frac{\pi}{2} \le \Im(z) \le \frac{\pi}{2}$ and $0 \le \Re(z) \le 1$ under the transformation $w = e^z$.

$$w = e^z = e^x \cdot e^{iy} = e^x + ie^x \sin y$$
$$\mu(x,y) = e^x \cos y \quad \text{and} \quad \nu(x,y) = e^x \sin y$$

Holding x fixed at $x = 1$, then $\mu^2 + \nu^2 = e^2(\cos^2 y + \sin^2 y) = e^2$. Thus, the image of the line segment $x = 1,\ -\pi/2 \le y \le \pi/2$ in the w-plane is the semicircle $\mu = e\cos y,\ \nu = e\sin y$, where $y \in [-\pi/2, \pi/2]$.

Similarly, holding x fixed at $x = 0$, then $\mu^2 + \nu^2 = \cos^2 y + \sin^2 y = 1$. Thus, the image of the line segment $x = 0,\ -\pi/2 \le y \le \pi/2$, in the w-plane is the semicircle $\mu = \cos y,\ \nu = \sin y$, where $y \in [-\pi/2, \pi/2]$.

Holding y fixed at $y = \pi/2$, then $\mu = 0,\ \nu = e^x$, where $0 \le x \le 1$. Then the image in the w-plane of the line segment $0 \le x \le 1,\ y = \pi/2$ is the line segment $\mu = 0,\ 1 \le \nu \le e$.

Similarly, holding y fixed at $y = -\pi/2$, then $\mu = 0,\ \nu = -e^x$, where $0 \le x \le 1$. Then the image in the w-plane of the line segment $0 \le x \le 1,\ y = -\pi/2$ is the line segment $\mu = 0,\ -e \le \nu \le -1$.