MATH 311 Final Summary

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1 Harmonic Functions

Definition 1.1. (Harmonic functions). Let $D \subset \mathbb{R}^2$ be a domain and let $h(x, y)$ be a continuous real-valued function with continuous partial derivatives. Then h is harmonic on D if h satisfies Laplace's equation, $h_{xx} + h_{yy} = 0$.

Theorem 1.1. *Let $f(z) = \mu + i\nu$ be analytic on D. Then μ and ν are harmonic on D.*

Example 1.2. Given $f(z) = 1/z$ is analytic on $D := \mathbb{C} \setminus \{0\}$,

$$f(z) = \frac{\bar{z}}{z\bar{z}} = \underbrace{\frac{x}{x^2 + y^2}}_{\mu} - i \underbrace{\frac{y}{x^2 + y^2}}_{\nu}$$

μ and ν are harmonic on D.

Example 1.3. Let $\mu(x, y) = 2x(1 - y)$. Find a real function $\nu(x, y)$ on $\mathbb{R}^2$ s.t. $f(z) = \mu + i\nu$ is entire (i.e., find the harmonic conjugate of μ).

$$\begin{aligned} \mu_x &= 2(1 - y) = \nu_y \implies \nu = 2y - y^2 + C(x) \implies \nu_x = C'(x) \\ -\mu_y &= 2x = \nu_x = C'(x) \implies C(x) = x^2 \implies \nu = 2y - y^2 + x^2 \\ f(z) &= 2(1 - y) + i(2y - y^2 + x^2) \text{ is entire.} \end{aligned}$$

2 Conformal Maps

Definition 2.1. Let D be a domain, $p \in D$, and $f : D \mapsto \mathbb{C}$. The function f is said to be conformal if it preserves angles at p. Furthermore, f is conformal on D if f is conformal $\forall p \in D$.

Theorem 2.1. *Suppose f is analytic on D, $p \in D$, and $f'(p) \neq 0$, $\forall p \in D$. Then f is conformal on D.*

Example 2.2. Let $f(z) = az + b,\ a \neq 0$. Then $f'(z) = a \neq 0$, so f is conformal on $\mathbb{C}$.

Example 2.3. Let $f(z) = z^2 \implies f'(z) = 2z \neq 0 \iff z \neq 0$, so f is conformal on $\mathbb{C} \setminus \{0\}$.

Theorem 2.4. *Let $D \subseteq \mathbb{C}$ be a domain and w be a non-constant function that is analytic at $p \in \mathbb{C}$. If*

$$w = w(p) + a_m(z - p)^m + (\textit{higher order terms})$$

where m is the smallest integer for which $f^{(n)}(p) \neq 0$, then the effect of the angle θ is $m\theta$.

Example 2.5. Let $f(z) = z^2$. Then $f'(z) = 2z + 2 \neq 0$, so f is conformal on $\mathbb{C} \setminus \{0\}$. Furthermore, $f''(z) = 2 \neq 0$ so the effect is $\theta \mapsto 2\theta$ at $z = 0$.

3 Contour Integrals

Definition 3.1. Let $z(t) : [a, b] \mapsto \mathbb{C}$ and $C : z([a, b])$ be the piecewise differentiable curve C parameterized by $z(t)$. Let $f(z)$ be a complex-valued function defined on C. Then the contour integral is defined as

$$\int_C f(z)\ dz := \int_a^b f(z(t))z'(t)\ dt$$

If $C = C_1 + C_2 + \cdots C_N$, then

$$\int_C f(z)\ dz = \sum_{i=1}^{N} \int_{C_i} f(z)\ dz$$

Example 3.1. Let $f(z) = \begin{cases} 1, & y > 0 \\ 4y, & y < 0 \end{cases}$, $z(t) = t + it^3 : [-1, 1] \mapsto \mathbb{C}$. Evaluate the integral $\int_C f(z)\ dz$.

$$\int_C f(z)\ dz = \int_{-1}^{0} 1(1 + 3it^2)\ dt + \int_0^1 4t^3(1 + 3it^2)\ dt = 4 + i$$

Definition 3.2. If a curve C is parameterized by $z(t) : [a, b] \mapsto C$, then the equivalent curve with opposite orientation is C^- parameterized by $z(t) := z(-t) : [-b, -a] \mapsto C^-$. Furthermore,

$$\int_C f(z)\ dz = -\int_{C^-} f(z)\ dz$$

Parameterizing straight lines: Let $P, Q \in \mathbb{C}$. Then $z(t) = Q + t(P - Q) : [0, 1] \mapsto C$ where $z(0) = Q$, and $z(1) = P$ is a straight line connecting P and Q.

4 Important Theorems

Theorem 4.1. *(Cauchy-Goursat theorem). Let C be a simple, closed curve and $f(z)$ be an analytic function on C and its interior. Then*

$$\int_C f(z) = 0$$

Example 4.2. By the Cauchy-Goursat theorem,

$$\int_{|z+1|=\pi} \frac{\sin z^8 - 14\cos^2(5z) + e^{e^z}}{e^{z^2 - 4z^3 + 4}} = 0$$

Theorem 4.3. *(ML inequality). Let $f(z) \leq M$ and $L = \int_C |dz|$. Then*

$$\left| \int_C f(z)\ dz \right| \leq ML$$

Example 4.4.

$$\begin{aligned} \left| \int_{|z|=2} \frac{z^3 + z - 2}{z^2 - 1}\ dz \right| &\leq \int_{|z|=2} \left| \frac{z^3 + z - 2}{z^2 - 1} \right|\ |dz| \\ &\leq \int_{|z|=2} \frac{|z|^3 + |z| + 2}{||z|^2 - 1|}\ |dz|, \quad \text{by the triangle inequality} \\ &= 4(2\pi \times 2) \\ &= 16\pi \end{aligned}$$

Example 4.5. Let C be a simple, closed curve containing 0 in its interior. Let $f(z) = z^n$. Then

$$\oint_C z^n\ dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & \text{otherwise} \end{cases}$$

Theorem 4.6. *(Cauchy integral formula). Let C be a simple, closed curve with point p in its interior. Let f be analytic on and inside C. Then*

$$f(p) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-p}$$

Furthermore,

$$f^{(n)}(p) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-p)^{n+1}}\ dz$$

Example 4.7. Evaluate

$$\int_C \frac{e^{iz}}{(z-1)^2}\ dz, \quad \text{where } C : |z+i| = 10$$

Note that $f(z) = e^{iz}$, $p = 1$, and $n = 1$.

$$\int_C \frac{e^{iz}}{(z-1)^2}\ dz = \frac{2\pi i}{1!}\frac{d}{dz}(e^{iz})|_{z=1} = -2\pi e^i$$

Example 4.8. Compute

$$\oint_C \frac{dz}{z^3(z+4)}$$

for (a) $|z| = 2$, (b) $|z+3| = 2$, (c) $|z| = 100$, and (d) $|z-100| = 1$.
(a) $f(z) = 1/(z+4)$, $p = 0$, and $n = 2$

$$\oint_C \frac{dz}{z^3(z+4)} = \frac{2\pi i}{2!}f''(0) = \frac{\pi i}{32}$$

(b) $f(z) = 1/z^3$, $n = 0$, and $p = -4$.

$$\oint_C \frac{dz}{z^3(z+4)} = \frac{2\pi i}{0!}f(-4) = -\frac{\pi i}{32}$$

(c) Add answers in (a) and (b) to get 0. (d) 0 by Cauchy-Goursat.

5 Standard Theorems in Complex Analysis

Theorem 5.1. *If f is analytic, f' is also analytic.*

Theorem 5.2. *(Liouville's theorem). The only bounded entire functions are constants.*

Theorem 5.3. *(Maximum modulus principle). Let f be analytic on domain $D \subset \mathbb{C}$ and fix $p \in D$. If $|f(z)| \leq |f(p)|$, $\forall z \in D$, then f is constant on D.*

Variant: Assume D is bounded, f is anaytic on D, and f extends to a continuous function on $\bar{D}$. If $f(z)$ non-constant on D, then $|f(z)|$ obtains its max on ∂D.

Theorem 5.4. *(Fundamental theorem of algebra). Let $p(z) \in \mathbb{C}$ be a polynomial of degree ≥ 1. Then $\exists r \in \mathbb{C}$ s.t. $p(r) = 0$.*

Theorem 5.5. *(Morera's theorem). Let $D \subseteq \mathbb{C}$ be a domain and $f(z)$ be a continuous complex-valued function on D. Suppose that*

$$\int_C f(z)\ dz = 0, \quad \forall C \subseteq D$$

Then $f(z)$ is analytic on D.

6 Power Series

Definition 6.1. (Radius of convergence). If both limits exist, they will yield the same R.

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

Example 6.1. Compute radius of convergence for $\sum_{n=1}^{\infty} \frac{n5^n}{i}(z+2i)^n$

$$R = \lim_{n\to\infty} \left| \frac{n5^n}{i} \times \frac{i}{(n+1)5^{n+1}} \right| = \frac{1}{5} \implies |z+2i| < \frac{1}{5}$$

Example 6.2. Compute radius of convergence for $\sum_{n=1}^{\infty} \frac{(2z+3i)^n}{n}$.

$$\sum_{n=1}^{\infty} \frac{[2(z+3i/2)]^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n} \left(z + \frac{3i}{2} \right)^n$$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|2^n/n|}} = \lim_{n\to\infty} \frac{1}{2(1/n)^{1/n}} = \frac{1}{2}$$

Example 6.3. Compute radius of convergence for $\sum_{n=0}^{\infty} \frac{2^n i}{z^n}$.

$$\lim_{n\to\infty} = \frac{1}{\sqrt[n]{2^n i}} = \frac{1}{2}$$

Converges if $|z| > 1/2$ and diverges if $|z| < 1/2$.

7 Laurent Series

Definition 7.1. A Laurent series centered at $p \in \mathbb{C}$ is the sum

$$\sum_{n=-\infty}^{\infty} c_n(z-p)^n = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n(z-p)^n$$

where $b_n = c_{-n}$ for $n \geq 1$ and $a_n = c_n$ for $n \geq 0$.

Theorem 7.1. *If $\exists r, R \in [0,\infty]$ st $\sum_{n=-\infty}^{\infty} c_n(z-p)^n$ defines an analytic function on $r < |z-p| < R$, then*

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} \quad \text{or} \quad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{or} \quad r = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

Theorem 7.2. *Assume that $0 \leq r < R \leq \infty$ and $f(z)$ is analytic on $r < |z-p| < R$. Then $f(z)$ is equal to a Laurent series on this annular region. Then the coefficients of the series are*

$$a_n = \frac{1}{2\pi i} \oint_{|z-p|=R'} \frac{f(z)}{(z-p)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \oint_{|z-p|=r'} f(z)(z-p)^{n-1}\, dz$$

Remark 7.3. *(Geometric series). If $\exists$singularity at $z = p$, then the Laurent expansion about p is*

$$\begin{cases} \dfrac{1}{1-cw} = \displaystyle\sum_{n=0}^{\infty} (cw)^n, & |w| < \frac{1}{|c|} \\ \dfrac{1}{1-c/w} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{c}{w}\right)^n, & |w| > |c| \end{cases}$$

where p lies on $|z| = c$ (think radius of convergence).

Example 7.4. Show that the Laurent series for

$$f(z) = \frac{5z}{z^2 + z - 6}$$

in the annular region $1 < |z - 1| < 4$ is given by

$$\sum_{n=1}^{\infty} \frac{2}{(z-1)^2} + \sum_{n=0}^{\infty} \frac{3(-1)^n}{4^{n+1}}(z-1)^n$$

Let $w = z - 1$ and observe that $z^2 + z - 6 = (z + 3)(z - 2) = (w + 4)(w - 1)$. Hence

$$\frac{5z}{z^2 + z - 6} = \frac{5(w+1)}{(w+4)(w-1)} = \frac{3}{w+4} + \frac{2}{w-1}$$

On $|w| > 1$, we have:

$$\frac{2}{w-1} = \frac{2}{w}\left(\frac{1}{1 - 1/w}\right) = \frac{2}{w}\sum_{n=0}^{\infty}\frac{1}{w^n} = \sum_{n=0}^{\infty}\frac{2}{w^{n+1}} = \sum_{n=1}^{\infty}\frac{2}{w^n} = \sum_{n=1}^{\infty}\frac{2}{(z-1)^n}$$

On $|w| < 4$ we have:

$$\frac{3}{w+4} = \frac{3}{4}\left(\frac{1}{1 + w/4}\right) = \frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}w^n = \sum_{n=0}^{\infty}\frac{3(-1)^n}{4^{n+1}}w^n = \sum_{n=0}^{\infty}\frac{3(-1)^n}{4^{n+1}}(z-1)^n$$

Thus on $1 < |w| = |z - 1| < 4$:

$$\frac{5}{z^2 + z - 6} = \sum_{n=1}^{\infty}\frac{2}{(z-1)^n} + \sum_{n=0}^{\infty}\frac{3(-1)^n}{4^{n+1}}(z-1)^n$$

Theorem 7.5. *(Riemann extension theorem). Suppose that $f(z)$ is both analytic and bounded on $0 < |z - p| < R$. Then $f(z)$ extends analytically to $z = p$.*

Theorem 7.6. *(Taylor series). If $b_n = 0, \forall n \geq 1$, then*

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(p)}{n!}(z-p)^n$$

Example 7.7.

$$\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}, \quad \cosh z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

Example 7.8. Compute the Taylor series for $f(z) = 1/(1 + z)^2$.

$$F(z) = \int\frac{dz}{(1+z)^2} = -\frac{1}{1+z} = -\sum_{n=0}^{\infty}(-1)^n z^n = \sum_{n=0}^{\infty}(-1)^{n+1}z^n$$

$$f(z) = F'(z) = \sum_{n=1}^{\infty}(-1)^{n+1}nz^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)z^n$$

Example 7.9. Compute the Taylor series for $f(z) = \log\frac{1+z}{1-z}$ on the branch where $\log 1 = 0$.

$$f(z) = \log(1+z) - \log(1-z) \implies f'(z) = \frac{1}{1+z} + \frac{1}{1-z} = \frac{2}{1-z^2} = 2\sum_{n=0}^{\infty}z^{2n}$$

$$f(z) = \int 2\sum_{n=0}^{\infty}z^{2n}\,dz = 2\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}$$

8 Isolated Singularities

Definition 8.1. Given an analytic function $f(z)$ on $0 < |z-p| < R \le \infty$,

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n (z-p)^n$$

Consider the point p. The function $f(z)$ has a

1. Removeable singularity at p if there is no principal part ($b_n = 0$).
2. Pole singularity at p if $\exists m \ge 1$ s.t. $b_n = 0, \ \forall n > m, \ b_n \ne 0$ for some n (truncated principal part).
3. Essential singularity at p if $b_n \ne 0, \ \forall n \ge 1$.

Example 8.1. (Removeable singularity). $f(z) = \sin z / z$.

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

Example 8.2. (Pole singularity). $f(z) = \cos z / z$.

$$\frac{\cos z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!}$$

Example 8.3. (Essential singularity). $f(z) = e^{1/z}$.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n}$$

Theorem 8.4. *(Classifying singularities).*

1. *p is a removeable singularity* $\iff$ $\lim_{z \to p} f(z) \in \mathbb{C}$ *exists.*
2. *p is a pole singularity* $\iff$ $\lim_{z \to p} f(z) = \infty$.
3. *p is an essential singularity* $\iff$ $\lim_{z \to p} f(z)$ *DNE.*

9 Zeroes of an Analytic Function

Definition 9.1. Let $f(z)$ be analytic on $|z-p| < R$. Assume $f(z) \ne 0$ on $|z-p| < R$ and $f(p) = 0$. By Taylor, $f(z) = a_n (z-p)^n + a_{n+1}(z-p)^{n+1} + \cdots$ where $a_n \ne 0$ for some $n \ge 1$. We say that p is a zero of $f(z)$ of order n.

Theorem 9.1. *Suppose $f(z)$ is analytic on a domain D and for some $p \in D$, we have $f^{(n)}(p) = 0, \ n \in \mathbb{Z}$. Then $f(z) = 0$ on D.*

Theorem 9.2. *Let $f(z) = \frac{b_M}{(z-p)^M} + \frac{b_{M-1}}{(z-p)^{M-1}} + \cdots + \frac{b_1}{z-p}, \ b_M \ne 0$ be analytic. Then f has a pole of order M at p $\iff$ $h := 1/f$ has a zero of order $M > 0$ at p.*

10 Residues

Theorem 10.1. *(Finding residues). Let $b_1 = Res_p f(z)$ be a residue of $f(z)$ at the point p.*

1. *Removeable singularities: $Res_p f(z) = 0$.*
2. *Simple pole: Find the Laurent series then find b_1 OR compute $b_1 = \lim_{z \to p} f(z)(z-p)$.*

3. *Pole of order m:*

$$b_1 = \lim_{z\to p} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-p)^m\right]$$

Example 10.2. $f(z) = e^{1/z}$.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \sum_{n=2}^{\infty}\frac{1}{n!z^n} \implies \text{Res}_0 e^{1/z} = 1$$

Example 10.3. $f(z) = e^{1/z^2}$

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \cdots \implies \text{Res}_0 e^{1/z^2} = 0$$

Example 10.4. Let $f(z) = 1/(z^2+1)$. Find the residue at $z = i$.
Set $g(z) = 1/(z+i) \implies g(i) = 1/(2i) \neq 0, f(z) = g(z)/(z-i)$. By Taylor,

$$g(z) = g(i) + g'(i)(z-i) + \frac{g''(i)}{2!}(z-i) + \cdots$$
$$f(z) = \frac{g(z)}{z-i} = \frac{g(i)}{z-i} + g'(i) + \frac{g''(i)}{2!}(z-i) + \cdots \implies \text{Res}_i f(z) = g(i) = \frac{1}{2i}$$

Example 10.5. $f(z) = \frac{e^z+1}{(z-\pi)^3}$. Compute $\text{Res}_\pi f(z)$.
Set $g(z) = e^z + 1,\ g(\pi) = e^\pi + 1 \neq 0,\ f(z) = g(z)/(z-\pi)^3$. Observe that π is a pole of order 3.

$$g(z) = g(\pi) + g'(\pi)(z-\pi) + \frac{g''(\pi)}{2!}(z-\pi)^2 + \cdots$$
$$f(z) = \frac{g(z)}{(z-\pi)^3} = \frac{g(\pi)}{(z-\pi)^3} + \frac{g'(\pi)}{(z-\pi)^2} + \frac{g''(\pi)}{2(z-\pi)} + \frac{g'''((\pi))}{3!} + \cdots$$
$$\implies \text{Res}_\pi f(z) = \frac{g''(\pi)}{2} = \frac{e^\pi}{2}$$

Theorem 10.6. *(Residue theorem). Let $f(z)$ be analytic on and inside a simple, closed curve C, except for a finite number of singular points $sing(f) = \{p_1, p_2, \ldots, p_k\}$ inside C and assume C has ccw orientation. Then*

$$\oint_C f(z)\ dz = 2\pi i \sum_{n=1}^{k} Res_{p_n} f(z)$$

Furthermore,

$$\sum_{p\in\{sing(f)\cup\infty\}} Res_p f(z) = 0 \implies -Res_\infty f(z) = Res_{w=0}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right)\right)$$
$$\sum_{p\in S} Res_p f(z) + Res_{other\ poles} f(z) = -Res_\infty f(z) = Res_{w=0}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right)\right)$$

where p is of a high multiplicity.

Example 10.7. Evaluate

$$\int_{|z|=2} \frac{z^9 e^{1/z}}{z^{10}+2}\ dz$$

Set $f(z) = z^9 e^{1/z}/(z^{10}+2)$. Then

$$\int_{|z|=2} f(z)\ dz = -2\pi i\ \text{Res}_\infty f(z) = 2\pi i\ \text{Res}_{w=0}\frac{1}{w^2}f\left(\frac{1}{w}\right)$$
$$\frac{1}{w^2}f\left(\frac{1}{w}\right) = \frac{1}{w^2}\frac{w^{-9}e^w}{w^{-10}+2} = \frac{1}{w^2}\frac{we^w}{1+2w^{10}} = \frac{e^w}{w(1+2w^{10})}$$
$$2\pi i\ \text{Res}_{w=0} = 2\pi i \lim_{w\to 0} w\frac{e^w}{w(1+2w^{10})} = 2\pi i$$

Example 10.8. Let $f(z) = \frac{1}{z^3(z+4)}$. Compute

$$\int_{|z|=2} f(z)\ dz \quad \text{and} \quad \int_{|z+2|=3} f(z)\ dz$$

By residue thm,

$$\left.\begin{array}{ll} z=-4: & \lim_{z\to -4}(z+4)f(z) = \frac{1}{(-4)^3} = -\frac{1}{64} \\ z=0: & \frac{1}{2!}\frac{d^2}{dz^2}\left[z^3 f(z)\right]_{z=0} = \frac{1}{64} \end{array}\right\} \implies \oint_{|z|=2} f(z)\ dz = 2\pi i\ \text{Res}_0 f(z) = \frac{\pi i}{32}$$

$$\oint_{|z+2|=3} f(z)\ dz = 2\pi i(\text{Res}_0 f(z) + \text{Res}_{-4} f(z)) = 0$$

Example 10.9. Compute $\text{Res}_0 f(z)$ where

$$f(z) = \frac{z^{12}+2z^6+1}{z^5(z^4 - \frac{5}{2}z^2 + 1)}$$

Use a geometric series:

$$f(z) = \frac{z^{12}+2z^6+1}{z^5\left(1 - [\frac{5}{2}z^2 - z^4]\right)} = \frac{z^{12}+2z^6+1}{z^5}\sum_{n=0}^{\infty}\left(\frac{5}{2}z^2 - z^4\right)^n = \frac{1}{z^5}\sum_{n=0}^{\infty}\left(\frac{5}{2}z^2 - z^4\right)^n + \text{analytic fn}$$

$$= \frac{1}{z^5}\left(1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4\right) + \text{another analytic fn} \implies \text{Res}_0 f(z) = -1 + \frac{25}{4} = \frac{21}{4}$$

Example 10.10. Let

$$f(z) = \frac{1}{z^{100}(z-i)^2}$$

Compute $\text{Res}_0 f(z)$.

$$\text{Res}_0 f(z) + \text{Res}_i f(z) = -\text{Res}_\infty f(z) := \text{Res}_{w=0}\frac{1}{w^2} f\left(\frac{1}{w}\right)$$

$$\text{Res}_{w=0}\frac{1}{w^2} f\left(\frac{1}{w}\right) = \text{Res}_{w=0}\frac{w^{100}}{(1-iw)^2} = 0$$

$$\text{Res}_i f(z) = \left(\frac{1}{z^{100}}\right)'(i) = -\frac{100}{i^{101}} = 100i$$

$$\implies \text{Res}_0 f(z) = -100i$$

Theorem 10.11. Suppose $f(z)$ is analytic on and inside a simple, closed ccw-oriented curve C, except on sing(f) inside C. Then

$$\oint_C \frac{f'(z)}{f(z)}\ dz = 2\pi i\left(\sum \text{mult}(z_k) + \sum \text{mult}(p_k)\right)$$

where $\text{mult}(z_k)$ is multiplicity of zeroes at z_k and $\text{mult}(p_k)$ is multiplicity of poles at z_k. We denote $N_{\text{zero},C}(f) := \sum \text{mult}(z_k)$ and $N_{\text{pole},C}(f) := \sum \text{mult}(p_k)$.

Definition 10.1. A function is meromorphic on domain $D \in \mathbb{C}$ if it has at worst isolated pole singularities on D, i.e., holomorphic on all of D except for a set of isolated points, which are poles of the function.

Theorem 10.12. (Argument principle). Given a simple, closed ccw-oriented curve C and f meromorphic in interior of C and analytic and non-zero on C,

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N_{\text{zero},C}(f) - N_{\text{pole},C}(f)$$

where LHS is the winding number of $f(C)$ about 0.

Example 10.13. Find the winding number of $f(z) = z^n,\ n \geq 1,\ C : |z| < \epsilon$.
By the argument principle, $\frac{1}{2\pi}\Delta_C \arg f(z) = n$.

Theorem 10.14. (Rouché's theorem). Given functions $f(z)$ and $g(z)$ analytic on and inside a simple, closed curve C and assume $|f(z)| > |g(z)|$ on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes (including multiplicity) inside C.

Example 10.15. Determine the roots in $f(z) = 2z^5 - 6z^2 + z + 1 = 0$ in the region $1 \leq |z| < 2$.

- Set $C = \{|z| = 1\},\ f(z) = -6z^2,\ g(z) = 2z^5 + z = 1$. Then $|f(z)| = 6 > 4 \geq |g(z)|$. By Rouché, f has 2 roots inside $|z| = 1$.
- Set $C = \{|z| = 2\},\ f(z) = 2z^5,\ g(z) = -6z^2 + z = 1$. Then $|f(z)| = 64 > 24 + 2 = 1 \geq |g(z)|$. By Rouché, f has 5 roots inside $|z| = 2$.
- Thus, $1 \leq |z| < 2$ has $5 - 2 = 3$ roots (including multiplicity).

11 Applications of Residues

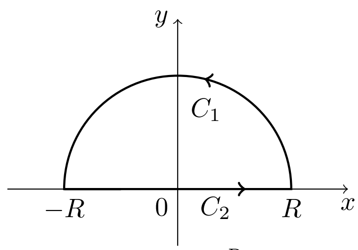
Example 11.1. (Cauchy principal values). Evaluate

$$\int_0^\infty f(x)\ dx, \quad f(x) = \frac{x^2 + 1}{(x^2 + \pi)(x^2 + \pi/2)}$$

Note that $\int_0^\infty f(x)\ dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\ dx$. First put

$$f(z) = \frac{z^2 + 1}{(x^2 + \pi)(z^2 + \pi/2)} = \frac{z^2 + 1}{(z + i\sqrt{\pi})(z - i\sqrt{\pi})(z + i\sqrt{\pi/2})(z - i\sqrt{\pi/2})}$$

Now consider the contour $C_R = C_1 \cup C_2$. Then we can write



$$\int_{C_R} f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz$$

By the ML inequality,

$$\left|\int_{C_2} f(z)\ dz\right| = \frac{R^2 + 1}{(R^2 - \pi)(R^2 - \pi/2)}(\pi R),\ R \to \infty,\ \text{RHS} \to 0$$

Then $\int_{C_R} f(z)\ dz = \int_{-R}^R f(x)\ dx$ by definition of a contour integral. By the residue theorem,

$$\lim_{R\to\infty}\int_{C_R} f(z)\ dz = 2\pi i\left[\text{Res}_{i\sqrt{\pi}} f(z) + \text{Res}_{i\sqrt{\pi/2}} f(z)\right] = \frac{2}{\sqrt{\pi}}(\sqrt{2} - 1)\left(\frac{\pi}{\sqrt{2} + 1}\right)$$

Therefore,

$$\int_0^\infty f(x)\ dx = \frac{1}{\sqrt{\pi}}(\sqrt{2} - 1)\left(\frac{\pi}{\sqrt{2} + 1}\right)$$

Example 11.2. (Improper integrals of trig fns). We wish to evaluate integrals of the form

$$\int_{-\infty}^\infty r(x)\begin{Bmatrix}\cos x \\ \sin x\end{Bmatrix}\ dx, \quad r(x) \text{ is a rational fn.}$$

Notice that

$$\int_{-\infty}^\infty r(x)e^{ix}\ dx = \int_{-\infty}^\infty r(x)\cos x\ dx + i\int_{-\infty}^\infty r(x)\sin x\ dx$$

Example 11.3. Evaluate
$$\int_0^\infty \frac{\cos 2x}{x^2+4}\ dx$$
We take the same contour as in Example 11.1. Then,
$$\Im\left(\lim_{R\to\infty}\int_{C_2} f(z)\ dz\right) = \int_{-\infty}^{\infty}\frac{\cos 2x}{x^2+4}\ dx$$
$$\left|\int_{C_1} f(z)\ dz\right| \le \left|\frac{e^{2iz}}{z^2+4}\right|\ |dz| \le \frac{1}{R^2+4}(\pi R) \xrightarrow{R\to\infty} 0$$
By residue theorem,
$$\int_{C_R}\frac{e^{2iz}}{z^2+4} = 2\pi i \text{Res}_{2i} f(z) = 2\pi i \lim_{z\to 2i}\frac{e^{2iz}}{z+2i} = \frac{\pi}{2e^4}$$
Therefore,
$$\int_0^\infty \frac{\cos 2x}{x^2+4}\ dx = \frac{\pi}{4}e^{-4}$$
Remark 11.4. CAUTION:
$$\lim_{R\to\infty}\left|\int_{C_1}\frac{\cos 2x}{x^2+4}\ dz\right| \ne 0, \quad \text{b/c } \cos 2x \text{ is not an odd fn.}$$
Example 11.5. (Definite integrals of trig fns).
$$\int_0^{2\pi} F(\cos\theta,\sin\theta)\ d\theta = \oint_{|z|=1} F\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} = 2\pi i \text{Res}_{p\in\{|z|<1\}}\frac{F}{iz}$$
CAUTION: Don't forget to divide by i when converting the sin term and computing the residue!

Example 11.6. Let $-1<a<1$ and $a\ne 0$. Evaluate
$$\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta}$$
We compute
$$\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta} = \int_{|z|=1}\frac{1}{1+a\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz} = \int_{|z|=1}\frac{dz}{z^2+2z/a+1}$$
Roots of denominator: $z=\left\{z_1=-\frac{1}{a}+\frac{1}{a}\sqrt{1-a^2}, z_2=-\frac{1}{a}-\frac{1}{a}\sqrt{1-a^2}\right\}$. Only z_1 is inside $|z|=1$.
$$\implies \int_0^{2\pi}\frac{d\theta}{1+a\cos\theta} = 2\pi i\frac{2}{ia}\text{Res}_{z_1}\frac{1}{z^2+2z/a+1} = \frac{2\pi}{\sqrt{1-a^2}}$$

12 Laplace Transform

Definition 12.1. The Laplace transform is defined as
$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$
Theorem 12.1. *(Inverse Laplace transform). Let $s\in\mathbb{C}$ be a complex variable and f be a real-valued function in domain $t\ge 0$. Assume f is piecewise continuous and exponentially bounded. Then*
$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)e^{st}\ ds$$

By residue theorem,

$$f(t) = \mathcal{L}^{-1}(F(s)) = \sum_{p:\ F(s)\ singular\ at\ p} Res_p(F(s)e^{st})$$

Furthermore,

$$f(t) = \mathcal{L}^{-1}(F(s)) = Res_{w=0} \frac{1}{w^2} e^{t/w} F\left(\frac{1}{w}\right)$$

Example 12.2. Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s+1)(s-1)^2}$$

Apply theorem 12.1.

$$\begin{aligned} F(s)e^{st} &= \frac{se^{st}}{(s+1)(s-1)^2} \\ \text{Res}_{-1} \frac{se^{st}}{(s+1)(s-1)^2} &= -\frac{e^{-t}}{4} \\ \text{Res}_{1} \frac{se^{st}}{(s+1)(s-1)^2} &= \frac{2\pi i}{1!} \frac{d}{ds} \left(\frac{se^{st}}{s+1} \right)\Big|_1 = \frac{(1+2t)e^t}{4} \\ f(t) &= -\frac{e^{-t}}{4} + \frac{(1+2t)e^t}{4} \end{aligned}$$