Heat Conduction Problems and the 1D Wave Equation

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Heat Equation	
Dirichlet BC $u_t = \alpha^2 u_{xx}$ $u(0,t) = 0$ $u(L,t) = 0$ $u(x,0) = f(x)$	$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin \frac{n\pi x}{L}$ $v(x) = 0$ $c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \ dx$
Neumann BC $u_t = \alpha^2 u_{xx}$ $u_x(0,t) = 0$ $u_x(L,t) = 0$ $u(x,0) = f(x)$	$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \cos \frac{n\pi x}{L}$ $v(x) = \frac{c_0}{2}$ $c_0 = \frac{2}{L} \int_0^L f(x) \ dx$ $c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \ dx$
Nonhomogeneous $u_t = \alpha^2 u_{xx}$ $u(0,t) = T_1$ $u(L,t) = T_2$ $u(x,0) = f(x)$	$u(x,t) = v(x) + w(x,t)$ $v(x) = \frac{T_2 - T_1}{L} x + T_1$ $w(x,0) = f(x) - v(x)$ $c_n = \frac{2}{L} \int_0^L (f(x) - v(x)) \sin \frac{n\pi x}{L} \ dx$
Forcing Term $u_t = \alpha^2 u_{xx} + q(x)$ $u(0,t) = T_1$ $u(L,t) = T_2$ $u(x,0) = f(x)$	$u(x,t) = v(x) + w(x,t)$ $v''(x) = -\frac{q(x)}{\alpha^2}$ $w(x,0) = f(x) - v(x)$ $c_n = \frac{2}{L} \int_0^L (f(x) - v(x)) \sin \frac{n\pi x}{L} \ dx$
Wave Equation	
General Case $u_{tt} = \alpha^2 u_{xx}$ $u(0,t) = 0$ $u(L,t) = 0$ $u(x,0) = f(x)$ $u_t(x,0) = g(x)$	$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi at}{L} + B_n \sin \frac{n\pi at}{L} \right) \sin \frac{n\pi x}{L}$ $u(x,t) = u_1(x,t) + u_2(x,t)$ Solve for the below two cases then add them.
Zero Initial Displacement $u_{tt} = \alpha^2 u_{xx}$ $u(0,t) = 0$ $u(L,t) = 0$ $u(x,0) = 0$ $u_t(x,0) = g(x)$	$u_1(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi at}{L} \sin \frac{n\pi x}{L}$ $A_n = 0$ $B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \ dx$
Zero Initial Velocity $u_{tt} = \alpha^2 u_{xx}$ $u(0,t) = 0$ $u(L,t) = 0$ $u(x,0) = f(x)$ $u_t(x,0) = 0$	$u_2(x,t) = \sum_{n=1}^{\infty} \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}$ $B_n = 0$ $A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \ dx$